

COMPLEX INTERSECTION OF REAL CYCLES IN REAL ALGEBRAIC VARIETIES AND GENERALIZED ARNOLD–VIRO INEQUALITIES

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§1. INTRODUCTION

1.1. The subject of the paper. Considering a real algebraic variety, we denote by $\mathbb{C}X$ its complex point set, by $\mathbb{R}X$ its real part being the fixed point set of the complex conjugation, $\text{conj}: \mathbb{C}X \rightarrow \mathbb{C}X$, and by \overline{X} the quotient $\mathbb{C}X/\text{conj}$.

Assuming that $\dim_{\mathbb{C}} \mathbb{C}X = d = 2n$ and that $\mathbb{C}X$ is a rational homology manifold, we obtain a \mathbb{Q} -valued quadratic form, ψ , induced on $H_d(\mathbb{R}X)$ by the inclusion homomorphism from the intersection form in $\mathbb{C}X$. More generally, we can require that $\mathbb{C}X$ is a \mathbb{Q} -homology manifold only in a neighborhood of $\mathbb{R}X$, or even more generally, that \overline{X} is a \mathbb{Q} -homology manifold in a neighborhood of $\mathbb{R}X$ (in the latter case, we consider the quadratic form on $H_d(\mathbb{R}X)$ induced from \overline{X}). The form ψ will be called *the complex intersection form* in $\mathbb{R}X$. Its analysis gives some information about the topology of $\mathbb{R}X$. For instance, in the case of a non-singular surface we obtain the Arnold inequalities, which I tried to extend to the case of singular varieties.

1.2. The Petrovskii and Arnold inequalities. Given a non-singular real algebraic curve $\mathbb{C}A \subset \mathbb{CP}^2$ of degree $2k$, recall the Petrovskii inequalities

$$(1-1) \quad \frac{3}{2}k(k-1) \leq p-n \leq \frac{3}{2}k(k-1)+1$$

where p and n denote the number of the components (called *ovals*) of $\mathbb{R}A$ lying inside even and, respectively, odd number of the other components. This result is related to the 16th Hilbert problem and gives, for instance, a negative answer to a particular question mentioned by Hilbert: are there non-singular real sextics, with the real part, $\mathbb{R}A$, having 11 ovals (11 is an upper bound for the number of ovals of a sextic provided by the Harnack theorem) which bound disjoint topological discs in \mathbb{RP}^2 .

To understand the nature of these inequalities, consider the double plane $\pi: X \rightarrow \mathbb{CP}^2$, branched along $\mathbb{C}A$. We can identify X with one of the real algebraic surfaces, $\mathbb{C}X^{\pm}$, defined by the equation $f(x, y, z) = \pm t^2$ in a quasi-homogeneous complex projective 3-space, where f is a real homogeneous polynomial of degree $2k$ defining $\mathbb{C}A$. Note that $\mathbb{R}X^{\pm}$ is mapped by π into the region $\mathbb{RP}_{\pm}^2 = \{[x : y : z] \in$

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$\mathbb{RP}^2 \mid \pm f(x, y, z) \geq 0\}$, one of which is orientable, and the other is not. Being free to vary the sign of f , we can assume that the orientable region is \mathbb{RP}_+^2 , then $\chi(\mathbb{R}X^+) = 2\chi(\mathbb{RP}_+^2) = 2(p - n)$, $\chi(\mathbb{R}X^-) = 2\chi(\mathbb{RP}_-^2) = 2(n - p + 1)$, and the Petrovskii inequalities can be formulated as the estimates

$$(1-2) \quad -\frac{1}{2}\chi(\mathbb{R}X^\pm) \leq \frac{1}{2}h^{1,1}(X) - 1$$

Note that the estimate (1-2), which belongs to Comessatti, historically precedes the Petrovskii inequalities, although its application to plane real curves in connection with Hilbert's problem was found later.

To prove (1-2) it is enough to use the Riemann–Hurwitz formula in combination with the formula for the signature of involution applied to the branched coverings $\mathbb{C}X^\varepsilon \rightarrow \overline{X}^\varepsilon$ and $\mathbb{C}X^\varepsilon \rightarrow \mathbb{CP}^2$, $\varepsilon = +$ or $\varepsilon = -$, which gives

$$(1-3) \quad \begin{aligned} 2\chi(\overline{X}^\varepsilon) - \chi(\mathbb{R}X^\varepsilon) &= \chi(X) = 2\chi(\mathbb{CP}^2) - \chi(\mathbb{C}A) = 4k^2 - 6k + 6 \\ 2\sigma(\overline{X}^\varepsilon) - \langle \mathbb{R}X^\varepsilon, \mathbb{R}X^\varepsilon \rangle_X &= \sigma(X) = 2\sigma(\mathbb{CP}^2) - \langle \mathbb{C}A, \mathbb{C}A \rangle_X = 2 - 2k^2, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_X$ stands for the intersection indices in X . Using that $b_1(X) = b_1(\overline{X}^\varepsilon) = 0$ and $\langle \mathbb{R}X^\varepsilon, \mathbb{R}X^\varepsilon \rangle_X = -\chi(\mathbb{R}X^\varepsilon)$, since the tangent and the normal bundle to $\mathbb{R}X^\varepsilon$ in $\mathbb{C}X^\varepsilon$ are anti-isomorphic, we have

$$(1-4) \quad b_2^+(\overline{X}^\varepsilon) = \frac{1}{2}(b_2^+(X) - 1) = p_g(X) = \frac{1}{2}(k-1)(k-2)$$

$$(1-5) \quad b_2^-(\overline{X}^\varepsilon) - \frac{1}{2}\chi(\mathbb{R}X^\varepsilon) = \frac{1}{2}(b_2^-(X) - 1) = \frac{1}{2}h^{1,1}(X) - 1 = \frac{3}{2}k(k-1),$$

where (1-5) shows that (1-2) and the Petrovskii inequality may be interpreted as $0 \leq b_2^-(\overline{X}^\varepsilon)$. The estimates (1-4) and (1-5) (and their corollaries (1-1), (1-2)) can be further enforced if we take into account existence of certain positive (or negative) square elements in $H_2(\overline{X}^\varepsilon)$. There are known two such enforcements. One is due to Comessatti, who used algebraic cycles to produce such elements, the other is due to Arnold [Ar], who used the components of $\mathbb{R}X^\varepsilon$ instead. More precisely, Arnold's inequalities appear from the estimates

$$(1-6) \quad \begin{aligned} c^\pm &\leq b_2^\pm(\overline{X}^\varepsilon) \\ c^\pm + c^0 &\leq b_2^\pm(\overline{X}^\varepsilon) + \delta(\mathbb{C}X^\varepsilon), \end{aligned}$$

where $\delta(\mathbb{C}X^\varepsilon) = \dim \ker(H_2(\mathbb{R}X^\varepsilon; \mathbb{R}) \rightarrow H_2(\mathbb{C}X^\varepsilon; \mathbb{R}))$ and c^+ , c^- , c^0 are the numbers of the connected oriented components of $\mathbb{R}X^\varepsilon$ with the negative, positive and zero Euler characteristic respectively (we refer to [Ar] for the original formulations and arguments, see also [V4] and subsection 5.1 below). The Smith theory gives an estimate $\delta(\mathbb{C}X^\varepsilon) \leq 1$ along with a certain information about the topology of $\mathbb{R}A$ in the case $\delta(\mathbb{C}X^\varepsilon) = 1$.

In [Zv], Zvonilov extended the Arnold inequalities to the non-singular curves $\mathbb{C}A$ of odd degrees applying a version of these inequalities to the curve obtained from $\mathbb{C}A$ by adding a line; this involved real curves with nodal singularities. The case of arbitrary plane real nodal curves was considered by Viro [V1]. In this case

the numbers of components, c^\pm , c^0 , in the Arnold inequalities must be replaced by the inertia indices, $\sigma_\pm(\psi_\varkappa)$, and the nullity, $\sigma_0(\psi_\varkappa)$, of the complex intersection form, ψ_\varkappa , on $H_2(\mathbb{R}X^\varepsilon)$; an estimate for $\delta(\mathbb{C}X^\varepsilon)$ is also required. Viro described this form in combinatorial terms and gave an estimate for $\delta(\mathbb{C}X^\varepsilon)$, which was later improved by Kharlamov and Viro. The current version of the Arnold–Viro inequalities for nodal curves together with a version of such inequalities for nodal surfaces is formulated in the survey of Kharlamov [Kh1] (the proofs, based on, or inspired by the ideas in the notes [V3], are reproduced in [F1]).

1.3. The results: generalized Arnold-Viro inequalities. Allowing $\mathbb{C}X$ to have more complicated singularities, one has to deal with three problems: to evaluate $b_d^\varepsilon(\overline{X})$, to give an estimate for $\delta(\mathbb{C}X^\varepsilon)$, and, the most important, to give a suitable combinatorial description of the complex intersection form. Theorem 8.1.1 allows to express $b_d^\varepsilon(\overline{X})$ in terms of $b_d^\varepsilon(\mathbb{C}X)$, provided $\mathbb{C}X$ is a \mathbb{Q} -homology manifold. Under a weaker assumption that \overline{X} is a \mathbb{Q} -homology manifold near $\mathbb{R}X$, Theorem 8.2.1 yields a formula for $b_d^\varepsilon(\overline{X})$ if $\mathbb{C}X$ has only ICIS (isolated complete intersection singularities), see §3. The estimates for $\delta(\mathbb{C}X^\varepsilon)$ (under various assumptions on $\mathbb{C}X$) are included in Appendix 1. An example of arrangements of hyperplanes considered in §6 shows that this estimate may admit, however, some improvement. And, finally, the integration formulae (2-3), (2-4), give a required combinatorial method of calculation of the form ψ_\varkappa , reducing the problem to evaluation of certain local invariant of singularities (the canonical quadratic form). To give a description of these local forms for arbitrary dimension of $\mathbb{C}X$, seems to be not an easy problem; I present in §5 some methods of calculation for the surface singularities. In particular, I justify such a method, announced in [F3], for the singularities, which appear by taking suspension over the real curve singularities.

As it often happens, the most general formulation of the result is not so convenient in applications as its special versions. Accordingly, we formulate below two such versions of the generalized Arnold-Viro inequalities, and refer to §3 and §4 for more general formulations.

Assume first that $\mathbb{C}A \subset \mathbb{C}P^d$, $d = 2n$, is a real hypersurface of even degree whose singular locus $\text{Sing}(\mathbb{C}A)$ contains only isolated singularities. Assume also that the double covering $X \rightarrow \mathbb{C}P^d$ branched along $\mathbb{C}A$ is a \mathbb{Q} -homology manifold. Denote by μ_x^\pm and μ_x^0 the ± 1 -inertia indices and the nullity of the Milnor form of a singularity at $x \in \text{Sing}(\mathbb{C}A) = \text{Sing } X$, and let μ^\pm and μ^0 denote the sum of μ_x^\pm and μ_x^0 , respectively, taken for all $x \in \text{Sing}(\mathbb{C}A)$. Put $\delta(\mathbb{C}X) = \dim \ker \text{in}$, where $\text{in}: H_d(\mathbb{R}X; \mathbb{Q}) \rightarrow H_d(\mathbb{C}X; \mathbb{Q})$ is the inclusion homomorphism.

Denote by $\mathbb{C}A^\tau$ a real non-singular perturbation of $\mathbb{C}A$ and by $X^\tau \rightarrow \mathbb{C}P^d$ the double covering branched along $\mathbb{C}A^\tau$. Then the complex intersection forms, ψ_\pm , arising on $H_d(\mathbb{R}X^\pm)$, satisfy the following inequalities

(1-7)

$$\begin{aligned} \sigma_{-\varkappa}(\psi_\pm) &\leq \frac{1}{2}(b_d^{-\varkappa}(X^\tau) - \kappa - \frac{1}{2}\mu^{-\varkappa}) + \min(0, \delta(\mathbb{C}X^\pm) - \sigma_0(\psi_\pm)) \\ \sigma_\varkappa(\psi_\pm) &\leq \frac{1}{2}(b_d^{-\varkappa}(X^\tau) + \chi(\mathbb{R}X^\pm) + \kappa - \mu^\varkappa) - 1 + \min(0, \delta(\mathbb{C}X^\pm) - \sigma_0(\psi_\pm)) \end{aligned}$$

where $\varkappa = (-1)^n$ and $\kappa \in \{0, 1\}$ is the mod 2 residue of n . These inequalities look certainly not complete unless we provide an estimate for $\delta(\mathbb{C}X)$. The following such an estimate is given in this paper

$$\delta(\mathbb{C}X) \leq (b_d(\mathbb{C}A; \mathbb{Z}/2) - \nu) + (n - 1)$$

where ν is the rank of the inclusion homomorphism $H_d(\mathbb{C}A; \mathbb{Z}/2) \rightarrow H_d(\mathbb{C}P^d; \mathbb{Z}/2)$.

In the following theorem we restrict ourselves with the case $d = 2$, but allow more general singularities of a reduced real curve $\mathbb{C}A \subset \mathbb{C}P^2$, requiring only that \overline{X}^+ (or \overline{X}^-) is a \mathbb{Q} -homology manifold in a neighborhood of $\mathbb{R}X^+$ (respectively, $\mathbb{R}X^-$); in particular, we impose no conditions on the imaginary singularities of $\mathbb{C}A$. Let $\mathfrak{p} = \frac{1}{2}(\mu^+ + \mu^0)$, that is the sum of the genera, $\mathfrak{p}_x = \frac{1}{2}(\mu_x^+ + \mu_x^0)$, of all the singular points $x \in \mathbb{C}A$. Put furthermore $\beta = \frac{1}{2}(\sum_x \mu_x^0)$, where the sum is taken for all $x \in \text{Sing}(\mathbb{C}A) - \text{Sing}(\mathbb{R}A)$. Denote by $\text{Sing}_0(\mathbb{C}A)$ the set of the *essential* singularities of $\mathbb{C}A$, that is, $\text{Sing}_0(\mathbb{C}A) = \{x \in \text{Sing}(\mathbb{C}A) \mid \mu_x = 0\}$, and let $\mathbb{C}A'' = \mathbb{C}A - \text{Sing}_0(\mathbb{C}A)$, $\mathbb{C}A' = \mathbb{C}A'' \cup \mathbb{R}A$.

1.3.1. Theorem. *Assume that \overline{X}^\pm is a \mathbb{Q} -homology manifold in a neighborhood of $\mathbb{R}X^\pm$. Then*

$$\begin{aligned}\sigma_+(\psi_\pm) &\leq \frac{1}{2}(k-1)(k-2) - \mathfrak{p} + \min\left(\frac{1}{2}(r-\nu), \frac{1}{2}\tilde{b}_0(\mathbb{C}A''), \tilde{b}_0(\mathbb{C}A'), \beta\right) \\ \sigma_+(\psi_\pm) + \sigma_0(\psi_\pm) &\leq \frac{1}{2}(k-1)(k-2) - \mathfrak{p} + \beta + (r-\nu) \\ \sigma_-(\psi_\pm) &\leq \frac{3}{2}k(k-1) + \frac{1}{2}\chi(\mathbb{R}X^\pm) - \frac{1}{2}\mu^- + \min\left(\frac{1}{2}(r-\nu), \frac{1}{2}\tilde{b}_0(\mathbb{C}A''), \tilde{b}_0(\mathbb{C}A'), \beta\right) \\ \sigma_-(\psi_\pm) + \sigma_0(\psi_\pm) &\leq \frac{3}{2}k(k-1) + \frac{1}{2}\chi(\mathbb{R}X^\pm) - \frac{1}{2}\mu^- + (r-\nu)\end{aligned}$$

where r is the number of irreducible components of $\mathbb{C}A$.

1.3.2. Corollary. *If $\mathbb{C}X^\pm$ has no essential singularities (i.e., is a \mathbb{Q} -homology manifold), then*

$$\begin{aligned}\sigma_+(\psi_\pm) &\leq \frac{1}{2}(k-1)(k-2) - \frac{1}{2}\mu^+ + \min(0, (r-\nu) - \sigma_0(\psi_\pm)) \\ &\quad - \frac{1}{2}\mu^+ + (r-\nu) \\ \sigma_-(\psi_\pm) &\leq \frac{3}{2}k(k-1) + \frac{1}{2}\chi(\mathbb{R}X^\pm) - \frac{1}{2}\mu^- + \min(0, (r-\nu) - \sigma_0(\psi_\pm)) \\ &\quad - \frac{1}{2}\mu^- + (r-\nu)\end{aligned}$$

1.4. Conventions. We denote by $b_k(X)$ the k -th Betti number of X and by $b_k(X; \mathbb{Z}/2)$ its $\mathbb{Z}/2$ -Betti-number, that is the rank of $H_k(X; \mathbb{Z}/2)$. Recall that \mathbb{Q} -homology manifolds have all the usual homology properties of manifolds (with the coefficients group \mathbb{Q}). We denote by $\langle \Gamma, \Delta \rangle_X$ the intersection index of oriented d -cycles, Γ, Δ , in X , provided X is an oriented compact $2d$ -dimensional \mathbb{Q} -homology manifold, (or, at least, such a manifold in a neighborhood of $\Gamma \cap \Delta$). We denote by $b_d^\pm(X)$ the inertia indices of the intersection form in X , when it is well-defined (i.e., for even d and X being a \mathbb{Q} -homology manifold as above), by $b_d^0(X)$ the nullity of this form (as X may have a non-empty boundary).

The prepositions \mathbb{C} , \mathbb{R} and a bar (e.g., $\mathbb{C}X$, $\mathbb{R}X$, \overline{X}) are used for the complex point sets, the real parts and the quotients associated to real algebraic varieties, as well as to conj-invariant subsets of such varieties. To simplify the notation, we identify $\mathbb{R}X$ with its image in \overline{X} . $\text{Sing}(X)$ denotes the singular locus of a complex

algebraic variety X ; moreover, for a subset $A \subset X$, we put $\text{Sing}(A) = \text{Sing}(X) \cap A$, for instance, $\text{Sing}(\mathbb{R}X) = \text{Sing}(\mathbb{C}X) \cap \mathbb{R}X$. Whenever the construction uses the metric in $\mathbb{C}X$ or $\mathbb{R}X$, we assume that it comes from the conj-invariant Fubini-Study metric in $\mathbb{CP}^N \supset \mathbb{C}X$.

Note that the standard construction of a semi-algebraic Whitney stratification of a real algebraic variety $\mathbb{C}X$, is readily conj-symmetric, which yields a suitable stratification of \overline{X} and provide us with a semi-algebraic stratification of $\mathbb{R}X$, which is obtained by taking intersection of $\mathbb{R}X$ with the strata of $\mathbb{C}X$. Taking the connected components of these intersections, we obtain a refinement, \mathfrak{S} , of the above stratification, that is used in §2.

Recall that the Euler characteristic with compact support, which is denoted by χ_c , is additive (see [GM2]) and can be used as a measure to integrate the appropriate functions, say, semi-algebraic functions on an algebraic variety (see [V2]). In this paper, we integrate functions which take constant values on the strata of \mathfrak{S} . Indeed, such an integration look more natural and obvious in the PL-category. Accordingly, we formulate and prove some results in 2.6 for polyhedra, keeping in mind that algebraic sets can be triangulated (see [Jo] for the modern proof and the further references on existence of a triangulation).

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§2. COMPLEX INTERSECTION OF REAL CYCLES IN A REAL ALGEBRAIC VARIETY

2.1. Inessential singularities and their canonical bilinear forms. Let us call a point of a complex d -dimensional algebraic variety an *inessential singularity*, or, briefly, a $\mathbb{Q}\mathbb{I}$ -point, if its link is a \mathbb{Q} -homology $(2d - 1)$ -sphere (that is a \mathbb{Q} -homology manifold having rational homologies of S^{2d-1}). We call a complex algebraic variety $\mathbb{Q}\mathbb{I}$ -variety if it has only inessential singular points. A $\mathbb{Q}\mathbb{I}$ -curve is obviously topologically non-singular, which easily implies that for any $\mathbb{Q}\mathbb{I}$ -variety X , its *topological singularity* (that is the set of points, $x \in X$, whose links is not homeomorphic to S^{2d-1}) has codimension ≥ 2 .

For a point, $x \in \mathbb{R}X$, of a real algebraic d -dimensional variety, we can choose a small regular compact conj-invariant neighborhood $\mathbb{C}U_x \subset \mathbb{C}X$ (for instance, ε -neighborhood in $\mathbb{CP}^N \supset \mathbb{C}X$, $0 < \varepsilon \ll 1$), put $\mathbb{C}M_x = \partial \mathbb{C}U_x$, and call x a $\overline{\mathbb{Q}\mathbb{I}}$ -point if \overline{M}_x is a rational homology $(2d - 1)$ -sphere. Note that a real $\mathbb{Q}\mathbb{I}$ -point is a $\overline{\mathbb{Q}\mathbb{I}}$ -point provided d is even. A real variety $\mathbb{C}X$ of even dimension will be called a $\overline{\mathbb{Q}\mathbb{I}}$ -variety if all its real points are $\overline{\mathbb{Q}\mathbb{I}}$ -points. It is not difficult to check that the topological singularity of $\mathbb{R}X$ for a $\overline{\mathbb{Q}\mathbb{I}}$ -variety $\mathbb{C}X$ has codimension ≥ 2 .

Given a Whitney stratified pseudo-manifold, Z , embedded smoothly (with respect to each stratum) in a smooth manifold, Y , we can define a *vector field tangent to Z* as a vector field in Y defined along Z , whose restrictions to the strata of Z are tangent to these strata.

Note that for any $x \in \mathbb{R}X$, there exists a tangent to $\mathbb{R}X$ vector field, ξ , defined along $\mathbb{R}M_x$, which is *transverse to $\mathbb{R}M_x$ and outward-directed*. Such ξ can be constructed by a stratified controlled lift [GM1, p.42] of the vector field $\frac{\partial}{\partial r}$ in \mathbb{R} with respect to the distance function $r: \mathbb{R}X \rightarrow \mathbb{R}$, $r(y) = \text{dist}(x, y)$. The vector field $i\xi$ (where $i = \sqrt{-1}$) is tangent to $\mathbb{C}M_x$ being normal to $\mathbb{R}X$ (and in particular to $\mathbb{R}M_x$).

Let γ and δ be $(d-1)$ -cycles representing some homology classes $[\gamma], [\delta] \in H_{d-1}(\mathbb{R}M_x)$, and $\delta_{i\xi}$ is the cycle in $\mathbb{R}M_x$ obtained by a small shift of δ in the direction of $i\xi$ (formally speaking, “a small shift” is carried by the local flow of a vector field $i\xi$; the problem of existence of the flow of stratified controlled vector fields is analyzed in [Ma]).

Then $\delta_{i\xi} \cap \mathbb{R}M_x = \emptyset$ and we can define a bilinear form $\lambda_x: H_{d-1}(\mathbb{R}M_x) \times H_{d-1}(\mathbb{R}M_x) \rightarrow \mathbb{Q}$, which assigns to the pair $([\gamma], [\delta])$ the linking number between γ and $\delta_{i\xi}$ in $\mathbb{C}M_x$. Note that λ_x is well defined and, in particular, is independent of the choice of ξ , because the set of all vector fields satisfying the requirement imposed on ξ is convex (with respect to the obvious linear homotopy). We call λ_x the *canonical quadratic form* associated to a real QI-point x (Proposition 2.2.3 justifies that λ_x is symmetric). One can treat λ_x as a local complex intersection form, due to the formula (being just a re-formulation of the definition of λ_x)

$$(2-1) \quad \lambda_x([\gamma], [\delta]) = \langle \text{Con}(\gamma), \text{Con}(\delta_\xi) \rangle_{\mathbb{C}U}$$

where $\text{Con}(\gamma)$, $\text{Con}(\delta_\xi)$ denote the cones over the cycles γ , δ_ξ in $\mathbb{C}U_x \cong \text{Con}(\mathbb{C}M_x)$.

Given a $\overline{\text{QI}}$ point $x \in \mathbb{R}X$ we define similarly a form $\overline{\lambda}_x$ on $H_{d-1}(\mathbb{R}M_x)$ assigning the linking number in \overline{M}_x between γ and the image of $\delta_{i\xi}$ in \overline{M}_x .

2.2. BASIC PROPERTIES OF THE CANONICAL FORMS OF THE INESSENTIAL SINGULARITIES

2.2.1. Proposition. *Assume that $x \in \mathbb{R}X$ is a QI-point and $d = \dim \mathbb{C}X$ is even. Then $\overline{\lambda}_x = 2\lambda_x$*

Proof. Let γ and δ be a pair of $(d-1)$ -cycles in $\mathbb{R}M_x$ (considered below also as cycles in $\mathbb{C}M_x$ and in \overline{M}) with the coefficients in \mathbb{Q} . Then $\gamma = \partial \overline{\sigma}$ for some d -chain $\overline{\sigma}$ in \overline{M}_x , and for the pull back, σ , of $\overline{\sigma}$ in $\mathbb{C}M_x$ we have obviously $\partial \sigma = 2\gamma$. Let $\delta_{i\xi}$ and $\delta_{-i\xi}$ denote the cycles in $\mathbb{C}M_x$ obtained from δ by shifts in the direction of $i\xi$ and $-i\xi$. Then $\langle \sigma, \delta_{i\xi} \rangle_{\mathbb{C}M_x} = \langle \sigma, \delta_{-i\xi} \rangle_{\mathbb{C}M_x}$, because conj preserves the orientation of M_x for even d and permutes $\delta_{i\xi}$ and $\delta_{-i\xi}$, and thus

$$(2-2) \quad 2\lambda_x([\gamma], [\delta]) = \langle \sigma, \delta_{i\xi} \rangle_{\mathbb{C}M_x} = \langle \overline{\sigma}, \overline{\delta}_{i\xi} \rangle_{\overline{M}_x} = \overline{\lambda}_x([\gamma], [\delta])$$

where $\overline{\delta}_{i\xi}$ is the image of $\delta_{i\xi}$ in \overline{M}_x . \square

For the next property note that the product of QI-varieties is again a QI-variety. Furthermore, if $\mathbb{C}X$ and $\mathbb{C}Y$ are *real* varieties and $x \in \mathbb{R}X$, $y \in \mathbb{R}Y$, then a homeomorphism between the real link, $\mathbb{R}M_z$, at $z = (x, y)$ and the join $\mathbb{R}M_x * \mathbb{R}M_y$ of the real links at x , y yields a canonical isomorphism $H_{p+q-1}(\mathbb{R}M_z) \cong H_{p-1}(\mathbb{R}M_x) \otimes H_{q-1}(\mathbb{R}M_y)$, where $p = \dim \mathbb{R}X$, $q = \dim \mathbb{R}Y$.

2.2.2. Proposition. *Assume that $\mathbb{C}X$ and $\mathbb{C}Y$ are real QI-varieties and $z \in \mathbb{R}X \times \mathbb{R}Y$. Then the canonical form λ_z is isomorphic to $(-1)^{pq} \lambda_x \otimes \lambda_y$, where λ_x , λ_y are the canonical forms at the points $x \in \mathbb{R}X$ and $y \in \mathbb{R}Y$, $z = (x, y)$.*

Combining Propositions 2.2.1 and 2.2.2, we obtain (for even p and q) $\overline{\lambda}_z \cong \frac{1}{2} \overline{\lambda}_x \otimes \overline{\lambda}_y$. Another corollary is that for a non-singular point $x \in \mathbb{R}X$, λ_z differs from λ_y only by the sign. In particular, if $x \in \mathbb{R}X$ belongs to a stratum $S \subset \mathbb{R}X$ of dimension p and codimension q , then $\lambda_x \cong (-1)^{pq + \binom{p+1}{2}} \lambda_S$, where λ_S denotes

the canonical form of the singularity at x in a real normal slice to S in $\mathbb{R}X$. By definition, such a slice is cut on $\mathbb{R}X$ by a real $(N-p)$ -dimensional plane in $\mathbb{R}P^N \supset \mathbb{R}X$, which is transversal to S . Since the strata S of our stratification, \mathfrak{S} , are connected, the form λ_S is independent of the choice of $x \in S$, due to the local triviality of $\mathbb{C}X$ along the Whitney strata.

Proof of Proposition 2.2.2. Choose $(p-1)$ -cycles, γ_1, δ_1 , in $\mathbb{R}M_x$ and $(q-1)$ -cycles, γ_2, δ_2 , in $\mathbb{R}M_y$ and denote by δ'_1, δ'_2 the cycles obtained from δ_1, δ_2 by small shifts along the canonical framings of $\mathbb{R}M_x$ and $\mathbb{R}M_y$. Then $\delta'_1 * \delta'_2$ is obtained by a similar shift of $\delta_1 * \delta_2$ in $\mathbb{C}M_z \cong \mathbb{C}M_x * \mathbb{C}M_y$. Using (2-1), we have

$$\begin{aligned} \lambda_z([\gamma_1 * \gamma_2], [\delta_1 * \delta_2]) &= \langle \text{Con}(\gamma_1 * \gamma_2), \text{Con}(\delta'_1 * \delta'_2) \rangle_{\mathbb{C}X \times \mathbb{C}Y} \\ &= \langle \text{Con}(\gamma_1) \times \text{Con}(\gamma_2), \text{Con}(\delta'_1) \times \text{Con}(\delta'_2) \rangle_{\mathbb{C}X \times \mathbb{C}Y} \\ &= (-1)^{pq} \langle \text{Con}(\gamma_1), \text{Con}(\delta'_1) \rangle_{\mathbb{C}X} \langle \text{Con}(\gamma_2), \text{Con}(\delta'_2) \rangle_{\mathbb{C}Y} \\ &= (-1)^{pq} \lambda_x([\gamma_1], [\delta_1]) \lambda_y([\gamma_2], [\delta_2]) \end{aligned}$$

□

2.2.3. Proposition. *The form λ_x as well as $\bar{\lambda}_x$ is symmetric, whenever it is well defined.*

Proof. First, note that $\text{lk}(\gamma, \delta_{i\xi}) = (-1)^d \text{lk}(\gamma, \delta_{-i\xi})$, where lk stands for the linking number in $\mathbb{C}M_x$. This is because conj changes the orientation for odd d , while preserving it for even d , and keeps γ fixed, interchanging $\delta_{i\xi}$ and $\delta_{-i\xi}$. Furthermore, we have obviously $\text{lk}(\gamma_{i\xi}, \delta) = \text{lk}(\gamma, \delta_{-i\xi})$ and thus

$$\begin{aligned} \lambda_x([\gamma], [\delta]) &= \text{lk}(\gamma, \delta_{i\xi}) = (-1)^d \text{lk}(\gamma, \delta_{-i\xi}) = (-1)^d \text{lk}(\gamma_{i\xi}, \delta) \\ &= \text{lk}(\delta, \gamma_{i\xi}) = \lambda_x([\delta], [\gamma]) \end{aligned}$$

In the case of the forms $\bar{\lambda}_x$, the arguments are the same, except that d may not be odd. □

Finally, we consider the natural generators of $H_d(\mathbb{R}X)$ for a real QI-variety $\mathbb{C}X$ of any dimension d , or $\overline{\text{QI}}$ -variety of even dimension, and give an integration formula for complex intersection of real cycles in $\mathbb{C}X$.

By a *component* of $\mathbb{R}X$ we mean the closure of a connected component of $\mathbb{R}X - \text{Sing}_{\text{top}}(\mathbb{R}X)$, where $\text{Sing}_{\text{top}}(\mathbb{R}X)$ is the topological singularity of $\mathbb{R}X$. As was mentioned in 2.1, the codimension of $\text{Sing}(\mathbb{R}X)$ is ≥ 2 for both QI- and $\overline{\text{QI}}$ -varieties, so the components of $\mathbb{R}X$ can be viewed as $(\mathbb{Z}/2)$ -cycles generating $H_d(\mathbb{R}X; \mathbb{Z}/2)$. Similarly, the orientable (outside $\text{Sing}_{\text{top}}(\mathbb{R}X)$) components of $\mathbb{R}X$, after we fix an orientation, represent the generators of $H_d(\mathbb{R}X)$.

Given a pair of oriented components Γ, Δ of $\mathbb{R}X$ and $x \in \Gamma \cap \Delta$, we put $\lambda_x(\Gamma, \Delta) = \lambda_x([\gamma], [\delta])$, where γ, δ are the cycles on $\mathbb{R}M_x$ cut by Γ and Δ .

2.2.4. Theorem. *Assume that Γ and Δ are oriented components of $\mathbb{R}X$ and $\Gamma \cap \Delta$ contains only QI-points. Then*

$$(2-3) \quad \langle \Gamma, \Delta \rangle_{\mathbb{C}X} = \int_{\Gamma \cap \Delta} \lambda_x(\Gamma, \Delta) d\chi(x)$$

If d is even and $\mathbb{R}X$ contains only $\overline{\text{QI}}$ -points, then

$$(2-4) \quad \langle \Gamma, \Delta \rangle_{\overline{\mathbb{C}X}} = \int_{\Gamma \cap \Delta} \bar{\lambda}_x(\Gamma, \Delta) d\chi(x)$$

2.2.5. Corollary. *If d is even and $\Gamma \cap \Delta$ contains only $\mathbb{Q}\mathbb{I}$ -points, then*

$$\langle \Gamma, \Delta \rangle_{\overline{X}} = 2\langle \Gamma, \Delta \rangle_{\mathbb{C}X}$$

Remark. Note that $\langle \Gamma, \Delta \rangle_{\mathbb{C}X}$ must vanish for any Γ, Δ if the dimension d is odd, however, λ_x may be non-trivial (see Example 2.4.3).

2.3. Proof of Theorem 2.2.4. To evaluate $\langle \Gamma, \Delta \rangle_{\mathbb{C}X}$ we follow the same approach as in non-singular case, i.e., shift Δ by the flow of $i\eta$, where η is an appropriate vector field, tangent to the strata of $\mathbb{R}X$. We suppose that η has a finite set, $\Sigma \subset \mathbb{R}X$, of singularities (zeros), so that $\Gamma \cap \Delta_{i\eta} = \Sigma$, where $\Delta_{i\eta}$ is a cycle obtained from Δ by a shift. Then $\langle \Gamma, \Delta \rangle_{\mathbb{C}X}$ is the sum of the local intersection indices, $\langle \Gamma, \Delta_{i\eta} \rangle_x$, at $x \in \Sigma$. For any $x \in \Sigma$, we express $\langle \Gamma, \Delta_{i\eta} \rangle_x$ in terms of the forms λ_x , provided the flow of η is positive (expanding) in the direction normal to the stratum, S , containing x .

To give a precise formulation of our assumption about η , we recall that the local triviality of Whitney stratified spaces along a stratum, S , (cf. [GM1, p. 37]), implies that there exists a chart $\text{ch}_x: U \rightarrow \mathbb{R}^p \times \mathbb{R}^{N-p}$ in a neighborhood $U \subset \mathbb{R}P^N$ of $x \in S \subset \mathbb{R}X$, mapping the points of $\mathbb{R}X$ to $\mathbb{R}^p \times N$ and, in particular, the points of the stratum S to $\mathbb{R}^p \times \{0\}$. Here N is a Whitney stratified space called *the normal slice* of S in $\mathbb{R}X$. We call η a *stratified expanding vector field with respect to a stratification*, \mathfrak{S} , of $\mathbb{R}X$, if for any $x \in \Sigma$ there exists a “product chart” as above, in which η splits into a direct sum, $\eta(x, y) = \eta_S(x) + \eta_N(y)$, where η_S is a vector field in \mathbb{R}^p (the components of η along S) and η_N is a field in \mathbb{R}^{N-p} , which is required to have a positive flow. Denote by $\text{ind}_x(\eta, \mathfrak{S})$ the index of η_S at x (if $S = \{x\}$ is a 0-dimensional stratum, then $\text{ind}_x(\eta, \mathfrak{S}) = 1$). A singularity of η_S will be called *elementary* if it is a standard non-degenerated singularity in some chart around x . A standard singularity of a vector field in \mathbb{R}^p is by definition represented either by “the identity” vector field, $\xi(x) = x$, or, by the “mirror reflection” field, $\xi(x_1, x_2, \dots, x_p) = (-x_1, x_2, \dots, x_p)$. If for a vector field, η , on $\mathbb{R}X$, its restrictions, η_S , to the strata, $S \in \mathfrak{S}$ have only elementary singularities, then η will be called an *elementary vector field*.

Theorem 2.2.4 follows from the following lemmas.

2.3.1. Lemma. *Given a real variety $\mathbb{C}X$ endowed with a Whitney stratification \mathfrak{S} , there exists an elementary stratified expanding vector field, η , with respect to \mathfrak{S} .*

2.3.2. Lemma. *Assume that η is like in lemma 2.3.1 and $S \subset \mathbb{R}X$ is a stratum of \mathfrak{S} . Then*

$$(2-5) \quad \sum_{x \in \Sigma \cap S} \text{ind}_x(\eta, \mathfrak{S}) = \chi_c(S)$$

2.3.3. Lemma. *Assume that Γ and Δ are like in Theorem 2.2.4, whereas η and S are like in Lemma 2.3.1. Then for any $x \in \Sigma \cap S$*

$$(2-6) \quad \langle \Gamma, \Delta_{i\eta} \rangle_x = \lambda_x(\Gamma, \Delta) \text{ind}_x(\eta, \mathfrak{S}),$$

Using (2-5)-(2-6), we obtain the formula (2-3) as follows

$$\begin{aligned}\langle \Gamma, \Delta \rangle_{\mathbb{C}X} &= \sum_{S \subset \Gamma \cap \Delta} \sum_{x \in S \cap \Sigma} \langle \Gamma, \Delta_{i\eta} \rangle_x = \sum_{S \subset \Gamma \cap \Delta} \sum_{x \in S \cap \Sigma} \lambda_x(\Gamma, \Delta) \text{ind}_x(\eta, \mathfrak{S}) \\ &= \sum_{S \subset \Gamma \cap \Delta} \lambda_S(\Gamma, \Delta) \chi_c(S) = \int_{\Gamma \cap \Delta} \lambda_x(\Gamma, \Delta) d\chi(x)\end{aligned}$$

where we put $\lambda_S(\Gamma, \Delta) = \lambda_x(\Gamma, \Delta)$ for $x \in S$, which makes sense, because $\lambda_x(\Gamma, \Delta)$ is independent of $x \in S$ due to the local triviality of $\mathbb{C}X$ along S and connectedness of S . The proof of (2-4) is analogous. \square

Proof of Lemma 2.3.1. Denote by Z_n the union of the strata of \mathfrak{S} of dimension ($\leq n$). We construct inductively a smooth vector field η_n on a neighborhood, $V_n \supset Z_n$ in $\mathbb{R}P^N \supset \mathbb{R}X$, so that η_n coincides with η_{n-1} on a neighborhood $V'_{n-1} \subset V_{n-1} \cap V_n$ of Z_{n-1} and satisfies the properties of a stratified expanding elementary vector field with respect to \mathfrak{S} (which include that η_n is tangent to $\mathbb{R}X$ along $\mathbb{R}X \cap V_n$, and have only elementary singularities, which form a set $\Sigma_n \subset Z_n$).

If $x \in Z_0$, then we let $\eta_0(x) = 0$ and define η_0 around x as the stratified controlled lift of the vector field $r^2 \frac{\partial}{\partial r}$ on \mathbb{R} , where $r: \mathbb{R}X \rightarrow \mathbb{R}$ is the distance from x . Given η_{n-1} , it is not difficult to extend it to n -strata, possibly, varying η_{n-1} outside some neighborhood of Z_{n-1} . For a generic such an extension, the singularities at $x \in \Sigma_n \setminus Z_{n-1}$ will be, obviously, non-degenerated. Furthermore, using an isotopy having support in a small neighborhood of x , it is not difficult to reduce a non-degenerated singularity to one of the two standard patterns making it elementary.

Finally, using a product chart, $\text{ch}_x: U \rightarrow \mathbb{R}^n \times \mathbb{R}P^{N-n}$, around $x \in Z_n - Z_{n-1}$, like above, we can locally extend our vector field, η_S , as a direct sum, $\eta_S + \eta_N$, to U . The field η_N around the origin in $\mathbb{R}P^{N-n}$ is constructed similarly to η_0 , using a stratified controlled lift of $r^2 \frac{\partial}{\partial r}$, with r being the distance function from the stratum S (i.e., from \mathbb{R}^n) in the chart ch_x . In particular, η_N has positive flow and is tangent to $\mathbb{R}X$ at the points of $\mathbb{R}X \cap U$. Patching together such local extensions via a partition of unity, we construct a field η_n defined, as is required, in a neighborhood, $V_n \supset Z_n$. Reducing the size of the domain V_n , if it is needed, we can make η_n have no zeros in $V_n - Z_n$. \square

Proof of Lemma 2.3.2. Let F_A denote the local flow of the vector field η restricted to $A \subset \mathbb{R}X$. The proof consists in applying the Lefschetz fixed point formula, in the form [GM2], to F_S , for a stratum, S , of \mathfrak{S} , which gives

$$(2-7) \quad \chi_c(S) = \sum_{x \in \Sigma \cap S} L_x(F_S)$$

where $L_x(F_S)$ is the local Lefschetz number of F_S at x , which coincides with $\text{ind}_x(\eta, \mathfrak{S})$. Since S is non-compact, we cannot directly apply the result of [GM2], and obtain (2-7) as the difference of the Lefschetz formulae applied to $F_{\text{Cl}S}$ and $F_{\partial S}$, where $\partial S = \text{Cl}S - S$

$$(2-8) \quad \begin{aligned}\chi(\text{Cl}S) &= \sum_{x \in \Sigma \cap \text{Cl}S} L_x(F_{\text{Cl}S}) \\ \chi(\partial S) &= \sum_{x \in \Sigma \cap \partial S} L_x(F_{\partial S})\end{aligned}$$

Recall that the Lefschetz formula in [GM2] requires *weak hyperbolicity* of a mapping (this property generalizes the Morse non-degeneracy condition and means, informally speaking, that a mapping can be representing near a fixed point as an expansion in one direction and a contraction in the complementary one), which is satisfied at least in the case of primitive stratified expanding vector fields with respect to \mathfrak{S} . To obtain (2-7), it is only left to notice that $L_x(F_{C \setminus S}) = L_x(F_{\partial S}) = \text{ind}_x(-\eta|_T)$, where $-\eta|_T$ is the restriction of $-\eta$ to the stratum, $T \subset \partial S$, of \mathfrak{S} containing x . These equalities follow from that the local Lefschetz number for germ of a mapping, as was defined in [GM2], does not change if we take its direct product with a germ of a contraction. \square

Proof of Lemma 2.3.3. Consider $x \in \Sigma \cap S$, where S is a stratum of \mathfrak{S} . If $\text{ind}_x(\eta, \mathfrak{S}) = 1$ and thus η_S is “the identity” vector field, in some chart around x , then we have $\langle \Gamma, \Delta_{i\eta} \rangle_x = \text{lk}_x(\gamma, \delta_{i\eta}) = \lambda_x(\gamma, \delta)$, so (2-6) holds. Here, like in the previous subsection, γ and δ are the cycles cut in $\mathbb{C}M_x$ by Γ and Δ , and lk_x is the linking number in $\mathbb{C}M_x$. If $\text{ind}_x(\eta, \mathfrak{S}) = -1$, then we should replace “the identity” vector field by “the mirror reflection”. But such a change effects on $\text{lk}_x(\gamma, \delta_{i\eta})$ and $\text{ind}_x(\eta, \mathfrak{S})$ as multiplication by -1 , so (2-6) still holds, since $\lambda_x(\gamma, \delta)$ is preserved, as it is independent of η . \square

2.4. QI^S singularities of hypersurfaces and their canonical forms. Consider an analytic reduced (i.e., not containing multiple factors) singularity $f: (\mathbb{C}^d, 0) \rightarrow (\mathbb{C}, 0)$, defining a germ of a hypersurface $\{f = 0\} = (\mathbb{C}A, 0) \subset (\mathbb{C}^d, 0)$. If $0 \in \mathbb{C}A$ is a QI -point, then f will be called *QI-singularity*. Similarly, by a $\overline{\text{QI}}$ -singularity we mean the complexification of a real analytic germ, f , such that $0 \in \mathbb{C}A$ is a $\overline{\text{QI}}$ -point. If the suspension, $f^S: (\mathbb{C}^{d+1}, 0) \rightarrow (\mathbb{C}, 0)$, $f^S(x_1, \dots, x_{d+1}) = f(x_1, \dots, x_d) - x_{d+1}^2$, over f , is QI -singularity, ($\overline{\text{QI}}$ -singularity), then we call f a QI^S -singularity (respectively, $\overline{\text{QI}}^S$ -singularity).

Note that the projection $\pi: (\mathbb{C}X, 0) \rightarrow (\mathbb{C}^d, 0)$ forgetting the last coordinate of the level set $\{f^S = 0\} = (\mathbb{C}X, 0) \subset (\mathbb{C}^{d+1}, 0)$ is a double covering branched along $(\mathbb{C}A, 0)$ and the singular locus, $\text{Sing}(\mathbb{C}X) = \text{Sing}(\mathbb{C}A)$, has codimension ≥ 2 . Denote by $\mathbb{C}B \subset \mathbb{C}^d$ a compact ε -ball ($0 < \varepsilon \ll 1$) around 0, let $\mathbb{C}S = \partial(\mathbb{C}B)$ and put $\mathbb{R}B_{\pm} = \{x \in \mathbb{R}B \mid \pm f(x) \geq 0\}$, $\mathbb{R}S_{\pm} = \mathbb{R}S \cap \mathbb{R}B_{\pm}$. Denote by V_i , $i = 1, \dots, s$, the closures of the connected components of $\mathbb{R}S \setminus \mathbb{R}A$ and call V_i *local partition regions of $\mathbb{R}A$ at 0*, putting $\text{sign}(V_i) \in \{+, -\}$ for the sign of f inside V_i . Put $\mathbb{C}M = \pi^{-1}(\mathbb{C}S)$, and orient $\mathbb{R}M = \pi^{-1}(\mathbb{R}S_+)$ (in the complement of $\text{Sing}(\mathbb{R}X)$), so that the restriction of the projection $\mathbb{R}M \rightarrow \mathbb{R}S$ to $\mathbb{R}M \cap (\mathbb{R}^d \times \mathbb{R}_+)$ preserves the orientation, whereas its restriction to $\mathbb{R}M \cap (\mathbb{R}^d \times \mathbb{R}_-)$ reverses ($\mathbb{R}S$ is oriented here as a boundary of $\mathbb{R}B \subset \mathbb{R}^n$). Similarly we orient $\pi^{-1}(\mathbb{R}S_-) \subset \mathbb{R}^d \times i\mathbb{R}$, making the restriction of π preserve the orientation on $\pi^{-1}(\mathbb{R}S_-) \cap (\mathbb{R}^d \times i\mathbb{R}_+)$, and reverse on $\pi^{-1}(\mathbb{R}S_-) \cap (\mathbb{R}^d \times i\mathbb{R}_-)$. With the inherited orientation, $\Theta_i = \pi^{-1}(V_i)$, $i = 1, \dots, s$, can be viewed as oriented cycles in $\mathbb{C}M = \pi^{-1}(\mathbb{C}S)$ (note that $\text{codim Sing}(\Theta_i) \geq 2$) and we denote by $[\Theta_i]$ their fundamental classes.

Given a QI^S -singularity f , we define a \mathbb{Q} -valued form λ^S on $\mathcal{H} = H^0(\mathbb{R}S \setminus \mathbb{R}A) \cong H_{d-1}(\mathbb{R}S, \mathbb{R}A \cap \mathbb{R}S)$ using the following version of the construction in 2.1. Denote by ξ a smooth vector field defined in a neighborhood of $\mathbb{R}S$ in \mathbb{R}^d , which is transverse to $\mathbb{R}S$ and outward-looking along it, being also tangent to $\mathbb{R}A$ at the points $x \in \mathbb{R}A$ (to construct ξ we use the stratified lifting theorem, like in 2.1). Denote by ξ^S the vector field in \mathbb{R}^{d+1} tangent to $\pi^{-1}(\mathbb{R}^d) \subset \mathbb{C}X$, obtained by lifting of ξ . Note that

the vector field $i\xi^S$ is tangent to $\mathbb{C}M$ and normal (in the standard metric of \mathbb{C}^{d+1}) to the set $R_M = p^{-1}(\mathbb{R}S) \subset \mathbb{C}M$ at points $x \in R_M$.

Denote by v_i , $i = 1 \dots, s$, the generators of \mathcal{H} represented by the *characteristic cochains* of V_i (equal to 1 on $V_i \setminus \mathbb{R}A$ and to 0 on the rest of $\mathbb{R}S \setminus \mathbb{R}A$), and put

$$\lambda^S(v_i, v_j) = \text{lk}(\Theta_i, \Theta'_j)$$

where lk is the linking number in $\mathbb{C}M$ and Θ'_j is obtained from Θ_j by a small shift in the direction of $i\xi^S$ (i.e., by the corresponding flow).

Note that the opposite choice of the orientation of \mathbb{R}^d (and, thus, of $\mathbb{R}S$), changes the orientation of Θ_i , but does not change the form λ^S . However, changing the sign of f , we interchange the summands, $\mathcal{H}_\pm = H^0(\mathbb{R}S_\pm \setminus \mathbb{R}A) \cong H_{d-1}(\mathbb{R}S_\pm, \mathbb{R}S_\pm \cap \mathbb{R}A)$, in the splitting $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$.

Denote by λ_\pm^S the restrictions of λ^S to \mathcal{H}_\pm . The construction of 2.1 applied to a cone-like neighborhood $\mathbb{C}U = \pi^{-1}(B)$ of $0 \in \mathbb{C}X$ defines the canonical form on $H_{d-1}(\mathbb{R}M)$ and it is not difficult to see that λ^{S+} is its pull back via the following product map

$$H^0(\mathbb{R}S_+ \setminus \mathbb{R}A) \cong H^0(\mathbb{R}S_+ \setminus \text{Sing}(\mathbb{R}A)) \xrightarrow{\pi^*} H^0(\mathbb{R}M \setminus \text{Sing}(\mathbb{R}M)) \cong H_{d-1}(\mathbb{R}M, \text{Sing}(\mathbb{R}M)) \cong H_{d-1}(\mathbb{R}M)$$

where the last isomorphism is due to that $\text{Sing}(\mathbb{R}M)$ has codimension ≥ 2 .

Similarly, we define a \mathbb{Q} -valued form $\bar{\lambda}^S$ on \mathcal{H}_+ , provided d is even and f is a $\overline{\text{QI}}^S$ -singularity. The arguments of Proposition 2.2.1 show that $\bar{\lambda}^S = 2\lambda^S|_{\mathcal{H}_+}$ if f is a QI^S -singularity and d is even.

Remark. By the Edmonds theorem, the fixed point set of a smooth involution on a Spin manifold gets certain *semi-orientation* (that is a pair of the opposite orientations) provided the involution preserves the orientation and the Spin structure. If f is an isolated singularity, then the Milnor fiber, $\mathbb{C}U_t = (f^S)^{-1}(t)$, $t \in \mathbb{R}$, of f^S is *Spin* and for $d \geq 2$ is simply connected, thus, if d is even, then conj preserves both the orientation and the Spin structure in $\mathbb{C}U_t$ and thus in $\mathbb{C}M_t = \partial(\mathbb{C}U_t)$. This endows $\mathbb{R}M \cong \mathbb{R}M_t$ with a semi-orientation. A slightly modified version of this construction can be applied to non-isolated singularities as well, and it is not difficult to show that the orientation of $\mathbb{R}M$, that we constructed above, coincides with a Spin semi-orientation.

Note that although the restrictions λ_\pm^S of λ^S are quadratic forms (isomorphic to the canonical quadratic forms of the singularities $f \pm x_{d+1}^2$), the bilinear form λ^S itself is not symmetric.

2.4.1. Proposition. *Assume that $\text{sign}(V_i) \neq \text{sign}(V_j)$. Then $\lambda^S(v_i, v_j) = -\lambda^S(v_j, v_i)$.*

The proof is analogous to that of Proposition 2.2.3. \square

This property of λ^S look more natural after changing the basis in $\mathcal{H}_\mathbb{C} = \mathcal{H} \otimes \mathbb{C}$. Namely, we put $\tilde{v}_k = iv_k$ (here $i = \sqrt{-1}$), if $\text{sign}(V_k) = -$, and $\tilde{v}_k = v_k$ if $\text{sign}(V_k) = +$, $k = 1, \dots, l$. The bilinear extension, $\lambda_\mathbb{C}^S$, of λ^S to $\mathcal{H}_\mathbb{C}$ is defined then by a self-adjoint matrix and the following “product formula” holds.

Assume that $h: (\mathbb{C}^{p+q}, 0) \rightarrow (\mathbb{C}, 0)$ is a product singularity, $h(x, y) = f(x)g(y)$, where $f: (\mathbb{C}^p, 0) \rightarrow (\mathbb{C}, 0)$ and $g: (\mathbb{C}^q, 0) \rightarrow (\mathbb{C}, 0)$ are QI^S -singularities. Then h is also a QI^S -singularity, since the germ $X_h = \{h^S = 0\}$ is obviously isomorphic to the quotient $(X_f \times X_g)/\theta$, where X_f and X_g are the germs defined by the equations

$f^S = 0$ and $g^S = 0$ and θ is the direct product of the deck transformations of the branched coverings $(X_f, 0) \rightarrow (\mathbb{C}^p, 0)$ and $(X_g, 0) \rightarrow (\mathbb{C}^q, 0)$. Denote by V'_1, \dots, V'_s and V''_1, \dots, V''_t the local partition regions for f and g respectively. Then such regions for h correspond to $V_{kl} = V'_k * V''_l$ under the natural homeomorphism $\mathbb{R}S_h \cong \mathbb{R}S_f * \mathbb{R}S_g$ (the objects like $\mathbb{R}S$, \mathcal{H} , $\mathcal{H}_{\mathbb{C}}$ and $\lambda_{\mathbb{C}}^S$ are marked by the subscripts f , g , h if they are associated to the corresponding singularities). Let $v'_k \in \mathcal{H}_f$, $v''_l \in \mathcal{H}_g$, $v_{kl} \in \mathcal{H}_h$ and $\tilde{v}'_k \in \mathcal{H}_{f\mathbb{C}}$, $\tilde{v}''_l \in \mathcal{H}_{g\mathbb{C}}$, $\tilde{v}_{kl} \in \mathcal{H}_{h\mathbb{C}}$ denote the corresponding bases. Consider the isomorphism $\mathcal{H}_{f\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{H}_{g\mathbb{C}} \cong \mathcal{H}_{h\mathbb{C}}$ which sends $\tilde{v}'_k \otimes \tilde{v}''_l$ to \tilde{v}_{kl} . The arguments analogous to that of Propositions 2.2.1–2.2.2 prove the following relation.

2.4.2. Proposition. $\lambda_{h\mathbb{C}}^S \cong \frac{1}{2}(-1)^{pq} \lambda_{f\mathbb{C}}^S \otimes \lambda_{g\mathbb{C}}^S$ \square

2.4.3. Example. The forms λ^S and $\lambda_{\mathbb{C}}^S$ for the identity function $f: (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$, $f(x) = x$, are defined by the matrices $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}$ in the bases $\{v_k\}$ and $\{\tilde{v}_k\}$ ($k = 1, 2$) respectively.

Using Proposition 2.4.2 we can determine the form $\lambda_{\mathbb{C}}^S$ for $f: (\mathbb{C}^d, 0) \rightarrow (\mathbb{C}, 0)$, $f(x_1, \dots, x_d) = x_1 \dots x_d$. Namely, let us mark the local partition regions of f by vectors $a = (a_1, \dots, a_d)$, $a_k \in \{+1, -1\}$ belonging to these regions. Let $v_a \in \mathcal{H}$, and $\tilde{v}_a \in \mathcal{H}_{\mathbb{C}}$ denote the basis elements representing the region containing a . Then (2-9)

$$\begin{aligned} \lambda_{\mathbb{C}}^S(\tilde{v}_a, \tilde{v}_b) &= (-1)^{\frac{d(d-1)}{2}} 2^{1-d} \text{sign}(b) i^{d(a,b)} = (-1)^{\frac{d(d-1)}{2}} 2^{1-d} \text{sign}(a) (-i)^{d(a,b)} \\ \lambda^S(v_a, v_b) &= (-1)^{\frac{d(d-1)}{2}} 2^{1-d} \text{sign}(b) i^{d(a,b) - a_- - b_-} \end{aligned}$$

where $\text{sign}(a) = a_1 \dots a_d$, and a_- , b_- denote the number of negative coordinates, whereas $d(a, b)$ the number of distinct coordinates in vectors a and b .

Finally, we introduce a relative version of the form λ^S , involving a $(\mathbb{Z}/2)$ -cycle $\Omega \subset \mathbb{R}^d$ of codimension 1, which contains 0 and have smooth simplices transversally intersecting $\mathbb{R}S$. Transversality yields a $(d-1)$ -cycle, $\Omega_S = \Omega \cap \mathbb{R}S$, in $\mathbb{R}S$. Its complement $\mathbb{R}S - \Omega_S$ splits, by the Alexander duality, in two regions distinguished by (mod 2)-linking number with Ω_S . Assuming that Ω_S does not intersect the interiors of V_i , V_j , we put $\lambda^S(v_i, v_j | \Omega) = \lambda^S(v_i, v_j)$, if the both V_i , V_j lie in the closure of one of the above two regions, and $\lambda^S(v_i, v_j | \Omega) = -\lambda^S(v_i, v_j)$ if they lie in the closures of distinct regions.

Similarly, we define $\bar{\lambda}^S(v_i, v_j)$ and $\bar{\lambda}^S(v_i, v_j | \Omega)$, for $v_i, v_j \in \mathcal{H}_+$, if f is a $\overline{\mathbb{Q}\mathbb{I}}^S$ -singularity.

2.5. The partition components of real hypersurfaces. Consider a non-singular real variety $\mathbb{C}P$, of dimension d and a real line bundle $\ell: \mathbb{C}\mathcal{L} \rightarrow \mathbb{C}P$, that is a line bundle supplied with an anti-linear involution, $\text{conj}_{\mathcal{L}}: \mathbb{C}\mathcal{L} \rightarrow \mathbb{C}\mathcal{L}$, commuting with ℓ and the complex conjugation in $\mathbb{C}P$. Let $\ell_{\mathbb{R}}: \mathbb{R}\mathcal{L} \rightarrow \mathbb{R}P$ denote the real part of ℓ , that is its restriction to the real parts of $\mathbb{C}\mathcal{L}$ and $\mathbb{C}P$.

The complex conjugation, $\text{conj}_{\mathcal{L}} \otimes \text{conj}_{\mathcal{L}}$, makes the square $\mathbb{C}\mathcal{L} \otimes \mathbb{C}\mathcal{L}$ a real bundle, whose real part is trivialized by choosing the direction of the “positive” ray $\{x \otimes x \mid x \in \mathbb{R}\mathcal{L}\}$ in each fiber. Thus, for a real (i.e., conjugation-equivariant) section $f: \mathbb{C}P \rightarrow \mathbb{C}\mathcal{L} \otimes \mathbb{C}\mathcal{L}$, the sign of $f(x)$ is well defined at the real points $x \in \mathbb{R}P$. We put $\mathbb{R}P_{\pm} = \mathbb{R}P_{\pm}(f) = \{x \in \mathbb{R}P : \pm f \geq 0\}$ and assume in what follows that the zero locus $\mathbb{C}A \subset \mathbb{C}P$ of f is a reduced hypersurface.

Denote by $\mathbb{C}X \subset \mathbb{C}\mathcal{L}$ the pull back of $f(\mathbb{C}P) \subset \mathbb{C}\mathcal{L}^{\otimes 2}$ via the mapping $\mathbb{C}\mathcal{L} \rightarrow \mathbb{C}\mathcal{L}^{\otimes 2}$, $x \mapsto x \otimes x$. The restriction $\pi = \ell|_{\mathbb{C}X}: \mathbb{C}X \rightarrow \mathbb{C}P$ is obviously a double covering branched along $\mathbb{C}A$.

Denote by W_j , $j = 1, \dots, m$, the closures of the connected components of $\mathbb{R}P - \mathbb{R}A$ and call W_j the *partition component* of $\mathbb{R}A$, putting $\text{sign}(W_i)$ for the sign of f inside W_i . Consider the class $\omega = w_1(\mathbb{R}P) + w_1(\ell_{\mathbb{R}}) \in H^1(\mathbb{R}P; \mathbb{Z}/2)$ and note that the restriction $\omega|_{W_j}$ vanishes if and only if $\Gamma_i = \pi^{-1}(W_i) \subset \mathbb{C}X$ is orientable (in the complement of $\text{Sing}(\mathbb{C}X)$). This is because $w_1(\mathbb{R}\mathcal{L}) = \ell^*(\omega)$ and the normal bundle to $\Gamma_i - \text{Sing}(\Gamma_i)$ is trivial (if $\text{sign}(W_i) = +$, then we consider the normal bundle in $\mathbb{R}\mathcal{L} \supset \Gamma_i$, otherwise we consider it in $i\mathbb{R}\mathcal{L} \supset \Gamma_i$).

Let W_1, \dots, W_l , $l \leq m$, be the components W_i with the restriction $\omega|_{W_j} = 0$, and put $\mathbb{R}P^\circ = \bigcup_{i=1}^l \text{Int } W_i$. Realize the homology class dual to ω by a $\mathbb{Z}/2$ -cycle in $\mathbb{R}P$ with smooth simplices and denote by Ω the union of the simplices. Then there exists an orientation of $\mathbb{R}\mathcal{L} - \ell_{\mathbb{R}}^{-1}(\Omega)$ which cannot be extended through the “walls” of $\ell_{\mathbb{R}}^{-1}(\Omega)$, and such an orientation is unique up to the natural action of $H^0(\mathbb{R}P; \mathbb{Z}/2)$. It is not difficult to choose Ω having support in $\mathbb{R}P - \mathbb{R}P^\circ$. If we fix such an orientation of $\mathbb{R}\mathcal{L} - \ell_{\mathbb{R}}^{-1}(\Omega)$, defined by Ω , and consider the orientation of $i(\mathbb{R}\mathcal{L} - \ell_{\mathbb{R}}^{-1}(\Omega))$ induced from it by the homeomorphism $\mathbb{R}\mathcal{L} \rightarrow i\mathbb{R}\mathcal{L}$, $x \mapsto ix$, then Γ_j , $j = 1, \dots, l$, become oriented cycles.

Given $x \in \mathbb{R}P$, we mark with a subscript x the objects $\mathbb{R}S$, λ^S , \mathcal{H} , etc., introduced in subsection 2.4, which are associated to the germ of f at x . Consider the *natural basis*, $w_i \in H^0(\mathbb{R}P - \mathbb{R}A)$, $i = 1, \dots, m$, represented by the characteristic cochains of $W_i \setminus \mathbb{R}A$ and let $w_i(x) \in \mathcal{H}_x$ denote the image of w_i under the inclusion homomorphism $H^0(\mathbb{R}P - \mathbb{R}A) \rightarrow H^0(\mathbb{R}S_x \setminus \mathbb{R}A)$. We call f a QI^S -*section* and $\mathbb{C}A$ a QI^S -*hypersurface* (respectively, a $\overline{\text{QI}}^S$ -*section* and a $\overline{\text{QI}}^S$ -*hypersurface*) if $\text{Sing}(\mathbb{C}A)$ contains only QI^S -singularities ($\overline{\text{QI}}^S$ -singularities); this is obviously equivalent to that $\mathbb{C}X$ is a QI -variety ($\overline{\text{QI}}$ -variety). If $\mathbb{C}A$ is a QI^S -hypersurface, then we define a bilinear \mathbb{Q} -valued *partition form* ϕ on $H^0(\mathbb{R}P^\circ)$ putting $\phi(w_i, w_j|\Omega) = \langle \Gamma_i, \Gamma_j \rangle_{\mathbb{C}X}$, $1 \leq i, j \leq l$ (here we keep the same notation, w_i , for the restriction of w_i to $H^0(\mathbb{R}P^\circ)$). Note that ϕ is well defined, in spite of the ambiguity in the choice of the orientation of $\mathbb{R}\mathcal{L} - \ell_{\mathbb{R}}^{-1}(\Omega)$.

2.5.1. Theorem. *Assume that $\mathbb{C}A \subset \mathbb{C}P$ is a QI^S -hypersurface. Then*
(2-10)

$$\begin{aligned} \phi(w_i, w_j|\Omega) &= \int_{W_i \cap W_j} \lambda_x^S(w_i(x), w_j(x)|\Omega) d\chi(x) & \text{if } 1 \leq i, j \leq l, i \neq j \\ \phi(w_i, w_i|\Omega) &= \int_{W_i \cap \text{Sing}(\mathbb{R}A)} \lambda_x^S(w_i(x), w_i(x)|\Omega) d\chi(x) \\ &+ (-1)^{\frac{d(d-1)}{2}} (2\chi_c(W_i \setminus \mathbb{R}A) + \chi_c((\mathbb{R}A \cap W_i) \setminus \text{Sing}(\mathbb{R}A))) & \text{if } 1 \leq i \leq l. \end{aligned}$$

Proof. It is not difficult to see that the orientations of the components Γ_i induce the same orientations of the cycles $\gamma_i = \mathbb{R}M_x \cap \Gamma_i$ as the components Θ_j in the previous section.

If $\text{sign}(W_i) = \text{sign}(W_j)$ are positive or negative, then the formulae of Theorem 2.5.1 follow from the formulae of Theorem 2.2.4. In the case of the opposite signs the proof is analogous. \square

Note that for even d the form ϕ is symmetric and splits into a direct sum, $\phi = \phi_+ \oplus \phi_-$, in $H^0(\mathbb{RP}^\circ) = H^0(\mathbb{RP}_+^\circ) \oplus H^0(\mathbb{RP}_-^\circ)$, where $\mathbb{RP}_\pm^\circ = \mathbb{RP}^\circ \cap \mathbb{RP}_\pm$. If f is a $\overline{\mathbb{Q}\mathbb{I}}^S$ -section, then we define a form $\bar{\phi}: H^0(\mathbb{RP}_+^\circ) \rightarrow \mathbb{Q}$ putting $\bar{\phi}(w_i, w_j) = \langle \Gamma_i, \Gamma_j \rangle_{\overline{\mathbb{X}}}$. Theorem 2.2.5 and the formula (2-4) then imply the following formula analogous to (2-10).

(2-11)

$$\begin{aligned} \bar{\phi}(w_i, w_j | \Omega) &= \int_{W_i \cap W_j} \bar{\lambda}_x^S(w_i(x), w_j(x) | \Omega) d\chi(x) && \text{if } 1 \leq i, j \leq l, i \neq j \\ \bar{\phi}(w_i, w_i | \Omega) &= \int_{W_i \cap \text{Sing}(\mathbb{R}A)} \bar{\lambda}_x^S(w_i(x), w_i(x) | \Omega) d\chi(x) \\ &+ (-1)^{\frac{d(d-1)}{2}} (4\chi_c(W_i \setminus \mathbb{R}A) + 2\chi_c((\mathbb{R}A \cap W_i) \setminus \text{Sing}(\mathbb{R}A))) && \text{if } 1 \leq i \leq l. \end{aligned}$$

Theorem 2.2.1 implies also that $\bar{\phi}(w_i, w_j) = 2\phi_+(w_i, w_j)$, provided $\mathbb{C}A$ is a $\mathbb{Q}\mathbb{I}^S$ -hypersurface and d is even.

2.6. Residue form. In this section we show how integration along the odd-dimensional strata in the formulae (2-10), (2-10), can be reduced to integration along their boundary. Assume that Q is a compact polyhedron of dimension d . Let us call a function $f: Q \rightarrow \mathbb{R}$ *constructible* if it is constant on the open simplices of some triangulation, \mathcal{T} , of Q . For such a function, we may consider its restriction to the link, $\text{Lk}_x(Q)$, of $x \in Q$ and define

$$\hat{f}(x) = \int_{\text{Lk}_x(Q)} f(y) d\chi(y)$$

The latter definition makes sense if the link $\text{Lk}_x(Q)$ is taken with respect to a sufficiently fine triangulation, say, the barycentric subdivision of any refinement of \mathcal{T} , containing x as a vertex (here, as above, f is constant of the simplices of \mathcal{T}). Alternatively, we may assume that $\text{Lk}_x(Q)$ is defined as the infinitesimal link of x (the direct limit of the usual links of x with respect to all triangulation containing x as a vertex, or, equivalently, the set of germs of PL -rays with the origin at x), and define the restriction $f|_{\text{Lk}_x(Q)}$ in the obvious way.

2.6.1. Lemma. *For any constructible function f on a compact polyhedron Q*

$$\int_Q \hat{f}(x) d\chi(x) = 0$$

Proof. This identity can be easily checked if f is a characteristic function of a closed simplex of \mathcal{T} . In general, f is a linear combination of such functions and the formula of the lemma follows from additivity of the integral. \square

2.6.2. Corollary. *Assume that the dimension, d , of Q is odd and Q_{d-1} denotes the union of the k -simplices, $k \leq d-1$, of \mathcal{T} . Then*

$$\int_Q f(x) d\chi(x) = \int_{Q_{d-1}} (f(x) - \frac{1}{2}\hat{f}(x)) d\chi(x)$$

Proof. This follows from Lemma 2.6.1 and from that $f(x) - \frac{1}{2}\hat{f}(x) = 0$ inside d -simplices of \mathcal{T} , for odd d . \square

Let Q_{sing} denote the *topological singularity* of Q , that is the set of points $x \in Q$ whose link, $\text{Lk}_x(Q)$, is not homeomorphic to $(d-1)$ -sphere, where $d = \dim Q$.

2.6.3. Corollary. *Assume that Q is a compact polyhedron of dimension d and $Q_{\text{sing}} \subset Q' \subset Q$ for a sub-polyhedron Q' . Then*

$$\begin{aligned} \int_{Q'} \chi(\text{Lk}_x(Q)) d\chi(x) &= 0, & \text{if } d \text{ is even} \\ \int_{Q'} \chi(\text{Lk}_x(Q)) d\chi(x) &= -2\chi_c(Q - Q'), & \text{if } d \text{ is odd} \end{aligned}$$

Proof. It follows from Lemma 2.6.1 applied to the constant function $f = 1$, since it gives $\hat{f}(x) = \chi(\text{Lk}_x(Q)) = 1 - (-1)^d$ for $x \in Q - Q_{\text{sing}}$. \square

Consider a real reduced hyper-surface $\mathbb{C}A$ in a real nonsingular d -dimensional variety $\mathbb{R}P$ and a partition component W_i defined as in the section 2.5. For $x \in \mathbb{R}P$, let $\chi_x(W_i) = \chi(W_i \cap \mathbb{R}S_x)$ where $\mathbb{R}S_x$ is an ε -sphere, $0 < \varepsilon \ll 1$, around x in $\mathbb{R}P^d$.

Applying Corollary 2.6.3 for $Q = W_i$ and $Q' = \mathbb{R}A \cap W_i$, we obtain

2.6.4. Corollary. *Assume that $\mathbb{C}A$ is like above. Then*

$$\begin{aligned} \int_{W_i \cap \text{Sing}(\mathbb{R}A)} \chi_x(W_i) d\chi(x) &= -\chi_c((\mathbb{R}A \cap W_i) \setminus \text{Sing}(\mathbb{R}A)) & \text{if } d \text{ is even} \\ \int_{W_i \cap \text{Sing}(\mathbb{R}A)} \chi_x(W_i) d\chi(x) &= -2\chi_c(W_i \setminus \mathbb{R}A) - \chi_c((\mathbb{R}A \cap W_i) \setminus \text{Sing}(\mathbb{R}A)) & \text{if } d \text{ is odd} \end{aligned}$$

\square

Using Corollary 2.6.2 and 2.6.4 we can rewrite the formulae (2-10) as follows (the “hat” over λ_x^S has below the same meaning as it has over f in Lemma 2.6.1).

2.6.5. Corollary. *Assume that $\mathbb{C}A$ is like in Theorem 2.5.1. Then*

$$\begin{aligned} \phi(w_i, w_j | \Omega) &= \int_{(W_i \cap W_j)_{k-1}} (\lambda_x^S(w_i(x), w_j(x) | \Omega) - \frac{1}{2} \hat{\lambda}_x^S(w_i(x), w_j(x) | \Omega)) d\chi(x) \\ & \quad \text{if } i \neq j \text{ and } k = \dim(W_i \cap W_j) \text{ is odd,} \\ \phi(w_i, w_i | \Omega) &= \int_{W_i \cap \text{Sing}(\mathbb{R}A)} (\lambda_x^S(w_i(x), w_i(x) | \Omega) - (-1)^{\frac{d(d-1)}{2}} \chi_x(W_i)) d\chi(x) \\ & \quad + (-1)^{\frac{d(d-1)}{2}} (1 + (-1)^d) \chi_c(W_i \setminus \mathbb{R}A) \quad \text{if } 1 \leq i \leq l. \end{aligned}$$

Here $(W_i \cap W_j)_{k-1}$ is constituted by the points of $W_i \cap W_j$ which belong to the strata of dimension $\leq k - 1$.

Note that Corollary 2.6.3 can be applied to $W_i \cap W_j$, since its topological singularity is contained in $(W_i \cap W_j)_{k-1}$. \square

The above formula becomes more simple if we assume that x is an *isolated* $\mathbb{Q}\mathbb{I}^S$ -singularity. In this case, the term $\chi_x(W_i)$ can be understood as the value $\chi_x(w_i(x), w_i(x))$ of the bilinear form, $\chi_x: \mathcal{H}_x \times \mathcal{H}_x \rightarrow \mathbb{Z}$, defined in the basis $v_1, \dots, v_s \in \mathcal{H}_x$ as follows

$$\begin{aligned} \chi_x(v_i, v_i) &= \chi(V_i) \\ \chi_x(v_i, v_j) &= \text{sign}(v_i) \frac{1}{2} \chi(V_i \cap V_j), \quad \text{if } i \neq j \end{aligned}$$

where $\text{sign}(v_i) = +1$ if V_i is a positive local partition component and $\text{sign}(v_i) = -1$ if negative. The term under the integral in Corollary 2.6.5 then becomes $q(w_i(x), w_j(x))$, where $q: \mathcal{H}_x \times \mathcal{H}_x \rightarrow \mathbb{Q}$ is the bilinear form

$$q = \lambda_x^S - (-1)^{\frac{d(d-1)}{2}} \chi_x$$

which will be called *the residue form*. We define also the relative form, $q(\cdot, \cdot | \Omega)$, of q using the same convention as in the subsection 2.4 for the form λ .

The formulae (2-10) can be rewritten as follows

2.6.6. Corollary. *Assume that $\mathbb{C}A \subset \mathbb{C}P$ is a hypersurface, like in Theorem 2.5.1, which have only isolated $\mathbb{Q}I^S$ -singularities. Consider a pair, W_i, W_j of partition components. Then*

$$(2-12) \quad \begin{aligned} \phi(w_i, w_j | \Omega) &= \sum_{x \in W_i \cap \text{Sing}(\mathbb{R}A)} q(w_i(x), w_j(x) | \Omega) \\ &+ (-1)^{\frac{d(d-1)}{2}} (1 + (-1)^d) \chi_c(W_i \setminus \mathbb{R}A) \delta_{ij} \quad \text{if } 1 \leq i \leq l. \end{aligned}$$

where δ_{ij} is the Kronecker symbol.

Similar arguments give an analogous formula for the form $\bar{\phi}$ in the case of $\overline{\mathbb{Q}I}^S$ -singularities. Namely, we consider the form $\bar{q} = \bar{\lambda}_x^S - 2(-1)^n \chi_x|_{\mathcal{H}_x^+}$ on \mathcal{H}_x^+ together with its relative form defined as usual. For $\mathbb{Q}I^S$ -singularities we have $\bar{q} = 2q_+$, where $q_{\pm} = q|_{\mathcal{H}_{\pm}}$. The formulae (2-11) can be stated then as follows.

2.6.7. Corollary. *Assume that $\mathbb{C}A \subset \mathbb{C}P$ is a $\overline{\mathbb{Q}I}^S$ -hypersurface, with isolated singularities, $d = 2n$ and $\text{sign}(W_i) = \text{sign}(W_j) = +$. Then*

$$(2-13) \quad \begin{aligned} \bar{\phi}(w_i, w_j | \Omega) &= \sum_{x \in W_i \cap \text{Sing}(\mathbb{R}A)} \bar{q}(w_i(x), w_j(x) | \Omega) \\ &+ 2(-1)^n \chi_c(W_i \setminus \mathbb{R}A) \delta_{ij} \quad \text{if } 1 \leq i \leq l. \end{aligned}$$

In case of $\mathbb{Q}I^S$ -hypersurface, for even d and $\text{sign}(W_i) \neq \text{sign}(W_j)$, we have obviously $\phi(w_i, w_j) = 0$, so, (2-12) implies that

$$0 = \phi(w_i, w_j) = \sum_{x \in \text{Sing}(\mathbb{R}A) \cap W_i \cap W_j} q(w_i(x), w_j(x) | \Omega)$$

It is not difficult to derive from the latter that all the terms of the above sum must vanish. This implies furthermore that $q(v_i(x), v_j(x)) = 0$, if $\text{sign}(V_i) \neq \text{sign}(V_j)$, for any isolated singularity $x \in \mathbb{R}A$, and thus $\lambda_x^S(v_i, v_j) = (-1)^{\frac{d(d-1)}{2}} \chi_x(v_i(x), v_j(x))$. We summarize it as follows.

2.6.8. Proposition. *Assume that $f: (\mathbb{C}^{2n+1}, 0) \rightarrow (\mathbb{C}, 0)$ is an isolated $\mathbb{Q}I^S$ -singularity and λ^S, q are the forms associated to it as above. Then*

- (1) $\lambda^S(v_i, v_j) = \text{sign}(v_i)(-1)^n \frac{1}{2} \chi(V_i \cap V_j)$ if $\text{sign}(V_i) \neq \text{sign}(V_j)$.
- (2) the form q splits into a direct sum $q = q_+ \oplus q_-$.

§3. GENERALIZED ARNOLD–VIRO
INEQUALITIES FOR COMPLETE INTERSECTIONS

3.1. The results. Given a real algebraic variety, $\mathbb{C}X$, of dimension d , we let $\delta(\mathbb{C}X) = \dim \ker(H_d(\mathbb{R}X; \mathbb{R}) \rightarrow H_d(\mathbb{C}X; \mathbb{R}))$, which is obviously equal to $\dim \ker(H_d(\mathbb{R}X; \mathbb{R}) \rightarrow H_d(\overline{X}; \mathbb{R}))$. As was mentioned in the introduction, the generalized Arnold–Viro inequalities are the estimates

$$\begin{aligned}\sigma_{\pm}(\psi) &\leq b_d^{\pm}(\overline{X}) \\ \sigma_{\pm}(\psi) + \sigma_0(\psi) &\leq b_d^{\pm}(\overline{X}) + \delta(\mathbb{C}X)\end{aligned}$$

for the inertia indices of the complex intersection form ψ in $H_d(\mathbb{R}X)$, being expressed in some suitable form. We can evaluate $b_d^{\pm}(\overline{X})$ in terms of $b_d^{\pm}(\mathbb{C}X)$, as it is done in Appendix B (Theorem 8.1.1). For instance, in the case of real complete intersection $\mathbb{Q}\mathbb{I}$ -varieties of dimension $d = 2n$, we obtain

$$\begin{aligned}(3-1) \quad b_d^{-\varkappa}(\overline{X}) &= \frac{1}{2}(b_d^{-\varkappa}(\mathbb{C}X) - \kappa) \\ b_d^{\varkappa}(\overline{X}) &= \frac{1}{2}(b_d^{\varkappa}(\mathbb{C}X) + \chi(\mathbb{R}X) - \kappa)\end{aligned}$$

where $\varkappa = (-1)^n$ and $\kappa = \frac{1}{2}(1 - \varkappa)$. Furthermore, for even n we have actually an estimate $\sigma_{\kappa}(\psi) = \sigma_+ \leq b_d^{\kappa}(\overline{X}) - 1$. To see it, note that the hyperplane section class, $H \in H_{d-2}(\mathbb{C}X)$, is anti-invariant with respect to conj (i.e., $\text{conj}_*(H) = -H$) and thus must be orthogonal to the image of $H_*(\mathbb{R}X)$ in the Lefschetz ring, $H_*(\mathbb{C}X)$. In particular, h^n vanishes on $H_d(\mathbb{R}X)$, where $h \in H^2(\mathbb{C}X; \mathbb{Q})$ is dual to H . On the other hand, h^n is conj^* -invariant for even n , and thus descends to a positive-square class in $H^d(\overline{X}; \mathbb{Q})$.

These results can be summarized in the following theorem.

3.1.1. Theorem. *Assume that $\mathbb{C}X$ is a real complete intersection $\mathbb{Q}\mathbb{I}$ -variety of dimension $d = 2n$ and $\varkappa = (-1)^n$, $\kappa = \frac{1}{2}(1 - \varkappa)$. Then*

$$\begin{aligned}(3-2) \quad \sigma_{-\varkappa}(\psi) &\leq \frac{1}{2}(b_d^{-\varkappa}(\mathbb{C}X) - \kappa) + \min(0, \delta(\mathbb{C}X) - \sigma_0(\psi)) \\ (3-3) \quad \sigma_{\varkappa}(\psi) &\leq \frac{1}{2}(b_d^{\varkappa}(\mathbb{C}X) + \chi(\mathbb{R}X) + \kappa) - 1 + \min(0, \delta(\mathbb{C}X) - \sigma_0(\psi))\end{aligned}$$

□

Evaluation of $b_d^{-\varkappa}(\mathbb{C}X)$ for an arbitrary $\mathbb{Q}\mathbb{I}$ -variety is beyond the scope of this paper. We only note that if such a variety, $\mathbb{C}X$, has only isolated complete intersection singularities (ICIS), then $b_d^{\pm}(\mathbb{C}X) = b_d^{\pm}(\mathbb{C}X^{\tau}) - \mu^{\pm}$, where $\mathbb{C}X^{\tau}$ denote a perturbation of $\mathbb{C}X$ (that is a non-singular real complete intersection obtained by a small variation of the equations defining $\mathbb{C}X$), and μ^{\pm} are the total Milnor numbers for $\mathbb{C}X$ defined like in 1.3. Evaluation of $b_d^{\pm}(\mathbb{C}X^{\tau})$, in the case of complete intersections, is also an easy problem, since the known Chern classes of $\mathbb{C}X^{\tau}$ determine obviously both $\chi(\mathbb{C}X^{\tau})$ and $\sigma(\mathbb{C}X^{\tau})$.

Under a weaker assumption that $\mathbb{C}X$ is a $\overline{\mathbb{Q}\mathbb{I}}$ -variety, a result similar to Theorem 3.1.1 needs less trivial calculations. We remove from $\mathbb{C}X$ a regular conj -symmetric compact regular neighborhood, $\mathbb{C}U_0$, of $\text{Sing}_0(\mathbb{C}X) - \text{Sing}_0(\mathbb{R}X)$ (the purely imaginary essential singularity) let $\mathbb{C}X' = \text{Cl}(\mathbb{C}X - \mathbb{C}U_0)$, and follow a similar approach applying it to the quotient \overline{X}' , which is a \mathbb{Q} -homology manifold. The result, which we present here only for the case of ICIS, is as follows.

3.1.2. Theorem. *Assume that $\mathbb{C}X$ is a complete intersection $\overline{\mathbb{Q}\mathbb{I}}$ -variety of dimension $d = 2n$, with only isolated singularities. Then*

(3-4)

$$\begin{aligned}\sigma_{-\kappa}(\overline{\psi}) &\leq \frac{1}{2}(b_d^{-\kappa}(\mathbb{C}X^\tau) - \kappa) - \mathfrak{p} + \min(\gamma, \beta + \delta(\mathbb{C}X') - \sigma_0(\psi)) \\ \sigma_{\kappa}(\overline{\psi}) &\leq \frac{1}{2}(b_d^{-\kappa}(\mathbb{C}X^\tau) + \chi(\mathbb{R}X) + \kappa - \mu^\kappa) - 1 + \min(\gamma - \beta, \delta(\mathbb{C}X') - \sigma_0(\psi))\end{aligned}$$

where $\beta = b_d(\partial\overline{U}_0)$, $\gamma = b_{d+1}(\overline{X})$, $\delta(\mathbb{C}X') = \dim \ker(H_d(\mathbb{R}X; \mathbb{R}) \rightarrow H_d(\overline{X}'; \mathbb{R}))$ and $\mathfrak{p} = \frac{1}{2}(\mu^{-\kappa} + \mu^0)$.

To make the inequalities in Theorems 3.1.1–3.1.2 usable, we need to complete them by estimating $\delta(\mathbb{C}X)$ and $\delta(\mathbb{C}X')$. Like in the case of the usual Arnold inequalities, such estimates come from the Smith theory (for the proof see Appendix A).

3.1.3. Proposition. *Assume that $\mathbb{C}X$ is a real projective algebraic variety of dimension d or a conj-invariant subset of such a variety. Then*

$$(3-5) \quad \delta(\mathbb{C}X) \leq \sum_{k=d+1}^{2d} b_k(\mathbb{C}X; \mathbb{Z}/2) - b_{d+1}(\overline{X})$$

In particular, if $\mathbb{C}X$ is a real complete intersection having only isolated singularities, and $d = 2n$, then

$$(3-6) \quad \delta(\mathbb{C}X) \leq b_{d+1}(\mathbb{C}X; \mathbb{Z}/2) + n - b_{d+1}(\overline{X})$$

Assume that $\mathbb{C}X \rightarrow \mathbb{C}P^d$, $d = 2n$, is a double covering branched along a real reduced hypersurface $\mathbb{C}A \subset \mathbb{C}P^d$. Then

$$(3-7) \quad \delta(\mathbb{C}X) \leq \sum_{k=d+1}^{2d-1} b_k(\mathbb{C}P^d, \mathbb{C}A; \mathbb{Z}/2) + (n-1) - b_{d+1}(\overline{X}),$$

If $\mathbb{C}A$ has in addition only isolated singularities, then

$$(3-8) \quad \delta(\mathbb{C}X) \leq (b_d(\mathbb{C}A; \mathbb{Z}/2) - \nu_d) + n - b_{d+1}(\overline{X})$$

Remarks.

- (1) The estimate (3-7) is better by 1 than (3-8) for $d = 2$.
- (2) If $\mathbb{C}X$ is a $\mathbb{Q}\mathbb{I}$ -variety and $d = \dim \mathbb{C}X$ is even, then $b_{d+1}(\overline{X}) = \frac{1}{2}b_{d+1}(\mathbb{C}X) = \frac{1}{2}b_{d-1}(\mathbb{C}X)$ (one can show it using that the mixed Hodge structure in $H^*(\mathbb{C}X)$ is pure, cf. 8.3). If moreover $\mathbb{C}X$ is a complete intersection, then $b_{d+1}(\overline{X}) = 0$.
- (3) The estimates (3-7) and (3-8) still hold if $\mathbb{C}P^d$ is replaced by any complete intersection real non-singular variety of dimension d , and a hypersurface $\mathbb{C}A$ is very ample (viewed as a divisor).

Proof of Theorem 3.1.2. The following Proposition 3.1.4 (proved in subsection 3.3) evaluates $b_d^\pm(\overline{X}')$ and $b_d^0(\overline{X}')$. To estimate $\delta(\mathbb{C}X')$ we use (3-5). The rest is analogous to the proof of Theorem 3.1.1. \square

3.1.4. Proposition. *Assuming that $\mathbb{C}X$ is like in Theorem 3.1.2, we have*

$$(3-9) \quad b_d^0(\overline{X}') = \beta - \gamma$$

$$(3-10) \quad b_d^{-\varkappa}(\overline{X}') = \frac{1}{2}(b_d^{-\varkappa}(\mathbb{C}X^\tau) - \kappa) - \mathfrak{p} + \gamma$$

$$(3-11) \quad b_d^{\varkappa}(\overline{X}') = \frac{1}{2}(b_d^{\varkappa}(\mathbb{C}X^\tau) + \chi(\mathbb{R}X) - \kappa - \mu^{\varkappa}) + \gamma - \beta$$

3.2. Two properties of the real ICIS (isolated complete intersection singularities). For the proof of Proposition 3.1.4 we need to use two results about real ICIS. The first one is a version of the Milnor Lemma for the quotients by the complex conjugation of a real ICIS. Given such a singularity $f: (\mathbb{C}^{d+k}, 0) \rightarrow (\mathbb{C}^k, 0)$, we put $\mathbb{C}U = \{f = 0\} \cap \mathbb{C}B \subset \mathbb{C}^{d+k}$, where $\mathbb{C}B$ is a compact ε -ball around zero, $0 < \varepsilon \ll 1$, and consider a small real deformation $f^\tau: (\mathbb{C}^{d+k}, 0) \rightarrow (\mathbb{C}^k, 0)$, $0 < |\tau| \ll \varepsilon$, of $f = f^0$ along with the corresponding deformation of $\mathbb{C}U$, denoted by $\mathbb{C}U^\tau$. We call f^τ (along with $\mathbb{C}U^\tau$) a *perturbation*, if $\mathbb{C}U^\tau$ is non-singular. It is well known (cf. [Lo], [D]) that $\mathbb{C}U^\tau$ is homotopy equivalent to a wedge of d -spheres, for any deformation, f^τ .

3.2.1. Lemma. *Assume that $\mathbb{C}U^\tau$ is a deformation of a cone-like compact neighborhood, $\mathbb{C}U$, for a real ICIS of dimension $d \geq 1$. Then the quotient \overline{U}^τ is homotopy equivalent to a wedge of d -spheres, provided $\mathbb{R}U^\tau \neq \emptyset$. If $\mathbb{R}U^\tau = \emptyset$, then \overline{U}^τ still has rational homology of a wedge of d -spheres.*

Proof. Note that for $d = 1$ the statement of the lemma is trivial, and the condition $\mathbb{R}U^\tau \neq \emptyset$ implies that \overline{U}^τ is simply connected for $d \geq 2$. Furthermore, $\pi_k(\overline{U}^\tau) = 0$ for $2 \leq k \leq d - 1$, since a generic mapping of S^k to \overline{U}^τ does not intersect $\mathbb{R}U^\tau$ and can be lifted to $\mathbb{C}U^\tau$. In addition, $H_k(\overline{U}^\tau; F) = 0$ for $k \neq d$, where F is a field of the characteristic $\neq 2$, for instance \mathbb{Q} or \mathbb{Z}/p , for a prime $p \neq 2$ (as it is well known that the projection $\mathbb{C}U^\tau \rightarrow \overline{U}^\tau$ induces an isomorphism between $H^k(\overline{U}^\tau; F)$ and the conj_* -invariant subspace of $H^k(\mathbb{C}U^\tau; F)$).

It follows also from the Smith sequence for conj (see 7.1) that $H_k(\overline{U}^\tau, \mathbb{R}U^\tau; \mathbb{Z}/2) = H_{k+1}(\overline{U}^\tau, \mathbb{R}U^\tau; \mathbb{Z}/2)$ for $k \geq d + 1$. Thus, $H_k(\overline{U}^\tau; \mathbb{Z}/2) = H_k(\overline{U}^\tau, \mathbb{R}U^\tau; \mathbb{Z}/2) = 0$, for $k \geq d + 1$. Furthermore, $H_d(\overline{U}^\tau)$ is torsion free by the universal coefficients formula, since $H_{d+1}(\overline{U}^\tau; \mathbb{Z}/p) = 0$ for any prime p , so it is left to apply the Whitehead theorem.

Remark. One can apply the same arguments in a more general setting, for instance, for non-isolated real singularities, and prove that \overline{U}^τ and of $\partial\overline{U} \cong \partial\overline{U}^\tau$ have the same connectedness properties as $\mathbb{C}U^\tau$ and $\partial\mathbb{C}U$ respectively, unless conj acts freely (the connectedness properties of $\mathbb{C}U^\tau$ and $\partial\mathbb{C}U$ can be found, e.g., in [Di, p.76]).

The next result is proven in Appendix B.

3.2.2. Theorem. *Assume that $\mathbb{C}U^\tau$ is a non-singular perturbation of a real ICIS of dimension, $d = 2n \geq 2$. Then*

$$(3-12) \quad b_d^{-\varkappa}(\overline{U}^\tau) + b_d^0(\overline{U}^\tau) = \frac{1}{2}(b_d^{-\varkappa}(\mathbb{C}U^\tau) + b_d^0(\mathbb{C}U^\tau))$$

$$(3-13) \quad b_d^{\varkappa}(\overline{U}^\tau) = \frac{1}{2}(b_d^{\varkappa}(\mathbb{C}U^\tau) + \chi(\mathbb{R}U^\tau) - 1)$$

3.3. Proof of Proposition 3.1.4. In this subsection we denote by $\mathbb{C}U \subset \mathbb{C}X$ a regular compact conj-invariant neighborhood of the whole $\text{Sing}(\mathbb{C}X)$ and by $\mathbb{C}U^\tau \subset \mathbb{C}X^\tau$ its non-singular perturbation, so that $\mathbb{C}X^\circ = \mathbb{C}X - \text{Int}(\mathbb{C}U)$ is identified with $\mathbb{C}X^\tau - \text{Int}\mathbb{C}U^\tau$. Note that \bar{X}° is obtained from a \mathbb{Q} -homology manifold \bar{X}' by removing several \mathbb{Q} -homology 4-discs, and thus, $b_d^\pm(\bar{X}') = b_d^\pm(\bar{X}^\circ)$ and $b_d^0(\bar{X}') = b_d^0(\bar{X}^\circ)$.

Furthermore, it follows from the long exact sequence of $(\bar{X}^\circ, \partial\bar{X}^\circ)$ that

$$b_d^0(\bar{X}^\circ) = b_{d-1}(\partial\bar{X}^\circ) - b_{d-1}(\bar{X}^\circ) + b_{d-1}(\bar{X}^\circ, \partial\bar{X}^\circ) - \tilde{b}_{d-2}(\partial\bar{X}^\circ)$$

since $b_{d-2}(\partial\bar{X}^\circ) = b_{d-2}(\bar{U}^\tau)$ vanish for $d > 2$ (cf. [Di, p. 76]). This implies (3-9), because, by the duality, $b_{d-1}(\bar{X}^\circ) = b_{d+1}(\bar{X}^\tau, \bar{U}^\tau) = b_{d+1}(\bar{X})$ and $b_{d-1}(\partial\bar{X}^\circ, \partial\bar{X}^\circ) = b_{d-1}(\bar{X}) + \tilde{b}_{d-2}(\partial\bar{X}^\circ)$, for $d \geq 2$.

The long exact sequence of $(\bar{X}^\tau, \bar{U}^\tau)$ yields

$$\begin{aligned} b_d(\bar{X}^\circ) &= b_d(\bar{X}^\tau, \bar{U}^\tau) = b_d(\bar{X}^\tau) - b_d(\bar{U}^\tau) + b_{d+1}(\bar{X}^\tau, \bar{U}^\tau) - b_{d+1}(\bar{X}^\tau) \\ (3-14) \quad &= b_d(\bar{X}^\tau) - b_d(\bar{U}^\tau) + \gamma \end{aligned}$$

Furthermore, $2b_d^{-\varkappa}(\bar{X}^\circ) = b_d(\bar{X}^\circ) - b_d^0(\bar{X}^\circ) - \varkappa\sigma(\bar{X}^\circ)$, where $\sigma(\bar{X}^\tau) = \sigma(\bar{X}^\circ) + \sigma(\bar{U}^\tau)$, and

$$2b_d^{-\varkappa}(\bar{X}^\circ) = b_d(\bar{X}^\tau) - b_d(\bar{U}^\tau) + \gamma - (\beta - \gamma) - \varkappa(\sigma(\bar{X}^\tau) - \sigma(\bar{U}^\tau))$$

Using (3-12) we obtain $b_d(\bar{U}^\tau) - \varkappa\sigma(\bar{U}^\tau) = 2b_d^{-\varkappa}(\bar{U}^\tau) + b_d^0(\bar{U}^\tau) = 2p - \beta$, which gives $2b_d^{-\varkappa}(\bar{X}^\circ) = (b_d(\bar{X}^\tau) - \varkappa\sigma(\bar{X}^\tau)) + 2\gamma - \beta - (2p - \beta)$ that is

$$(3-15) \quad b_d^{-\varkappa}(\bar{X}^\circ) = b_d^{-\varkappa}(\bar{X}^\tau) - p + \gamma$$

The relation (3-3) applied to $\mathbb{C}X^\tau$ gives (3-10). Subtracting (3-9) and (3-15) from (3-14), we obtain

$$(3-16) \quad b_d^\varkappa(\bar{X}^\circ) = b_d^\varkappa(\bar{X}^\tau) - b_d^\varkappa(\bar{U}^\tau) + \gamma - \beta$$

Finally, (3-13) gives $\chi(\mathbb{R}X^\tau) - \chi(\mathbb{R}X) = \tilde{\chi}(\mathbb{R}U^\tau) = 2b_d^\varkappa(\bar{U}^\tau) - \mu^\varkappa$, which together with (3-16) and (3-3) applied to $\mathbb{C}X^\tau$ implies the last identity (3-11).

§4. GENERALIZED ARNOLD–VIRO INEQUALITIES FOR REAL ALGEBRAIC SURFACES

4.1. Generalized Arnold–Viro inequalities for curves. Let ρ denote the number of real branches at 0 of the zero locus, $(\mathbb{C}A, 0) = \{f = 0\}$, of a real singularity $f: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$. Let us call f (as well as $(\mathbb{C}A, 0)$) a *dot singularity* if $\rho = 0$. A dot singularity can be *positive*, if f is positive around 0 on \mathbb{R}^2 , and *negative* otherwise.

Given a real even $\mathbb{Q}\mathbb{I}^S$ -curve, $\mathbb{C}A$, on a non-singular real surface, $\mathbb{C}P$, we consider the partition form, ϕ , defined in 2.5. Recall that $\mathbb{C}A$ is the zero locus of a real section, f , of $\mathbb{C}\mathcal{L}^{\otimes 2}$ for some real line bundle, $\ell: \mathbb{C}\mathcal{L} \rightarrow \mathbb{C}P$, and f being fixed defines a splitting $\phi = \phi_+ \oplus \phi_-$. Recall also that the double covering, $\pi: \mathbb{C}X \rightarrow \mathbb{C}P$, branched along $\mathbb{C}A$ is the restriction of ℓ to $\mathbb{C}X \subset \mathbb{C}\mathcal{L}$. Denote by K the canonical

class of $\mathbb{C}P$ and by L the divisor class of the line bundle $\ell: \mathbb{C}\mathcal{L} \rightarrow \mathbb{C}P$, introduced in 2.5.

Assume that f admits a real perturbation, f^τ , so that the corresponding perturbation, $\mathbb{C}A^\tau$, of $\mathbb{C}A$ is non-singular. Let $\pi^\tau: \mathbb{C}X^\tau \rightarrow \mathbb{C}P$ denote the corresponding perturbation of π . For an ε -neighborhood, $\mathbb{C}B \subset \mathbb{C}P$, of $\text{Sing}(\mathbb{C}A)$, we put $\mathbb{C}U = \pi^{-1}(\mathbb{C}B)$, $\mathbb{C}U^\tau = (\pi^\tau)^{-1}(\mathbb{C}B)$, assuming that $0 < |\tau| \ll \varepsilon \ll 1$. We consider moreover an ε -neighborhood $\mathbb{C}B_0$ of the essential imaginary singular locus, $\text{Sing}_0(\mathbb{C}A) \setminus \text{Sing}(\mathbb{R}A)$, and put similarly $\mathbb{C}U_0 = \pi^{-1}(\mathbb{C}B_0)$, $\mathbb{C}U_0^\tau = (\pi^\tau)^{-1}(\mathbb{C}B_0)$. Put furthermore $\mathbb{C}P^\circ = \text{Cl}(\mathbb{C}P - \mathbb{C}B)$, $\mathbb{C}A^\circ = \mathbb{C}A \cap \mathbb{C}P^\circ$, $\mathbb{C}X^\circ = \text{Cl}(\mathbb{C}X - \mathbb{C}U)$ and define similarly $\mathbb{C}P'$, $\mathbb{C}A'$, $\mathbb{C}X'$, (respectively, $\mathbb{C}P''$, $\mathbb{C}A''$, $\mathbb{C}X''$) removing from $\mathbb{C}P$, $\mathbb{C}A$, $\mathbb{C}X$ ε -neighborhoods of the essential imaginary singular loci (respectively, neighborhoods of the whole essential singular loci).

Recall that $\mu^\pm = b_2^\pm(\mathbb{C}U^\tau)$, $\mu^0 = b_2^0(\mathbb{C}U^\tau)$, $\mathfrak{p} = \frac{1}{2}(\mu^+ + \mu^0)$. The imaginary essential singularities of $\mathbb{C}A$ split into complex-conjugated pairs; denote by $\alpha_{\text{Im}}^{(0)}$ the number of these pairs. Consider a very good real resolution $\mathbb{C}X^{\text{res}} \rightarrow \mathbb{C}X$ of $\mathbb{C}X$, denote its exceptional divisor by $\mathbb{C}E$, and put $\beta = b_1(\bar{E})$.

Denote by ν (ν') the rank of the inclusion homomorphism from $H_2(\mathbb{C}A; \mathbb{Z}/2)$ (respectively, from $H_2(\mathbb{C}A'; \mathbb{Z}/2)$) to $H_2(\mathbb{C}P; \mathbb{Z}/2)$, and let t_2 denote the rank of $\mathbb{Z}/2$ -torsion in $H_1(\mathbb{C}P)$.

4.1.1. Theorem. *Assume that $\mathbb{C}A$ is a $\overline{\text{QI}}^S$ curve. Then*

$$(4-1) \quad \begin{aligned} \sigma_+(\bar{\phi}) &\leq b_2^+(\mathbb{C}P) + \frac{1}{2}L.(K + L) - \mathfrak{p} + t_2 \\ &\quad + \frac{1}{2} \min(\tilde{b}_0(\mathbb{C}A''), \tilde{b}_0(\mathbb{C}A) + 2\beta, b_2(\mathbb{C}A) - \nu) \end{aligned}$$

$$(4-2) \quad \begin{aligned} \sigma_+(\bar{\phi}) + \sigma_0(\bar{\phi}) &\leq b_2^+(\mathbb{C}P) + \frac{1}{2}L.(K + L) - \mathfrak{p} + 2t_2 + b_1(\mathbb{C}P) + (b_2(\mathbb{C}A') - \nu') + \beta \\ &\quad + \max(0, 3\alpha_{\text{Im}}^{(0)} - 1) \end{aligned}$$

$$(4-3) \quad \begin{aligned} \sigma_-(\bar{\phi}) &\leq b_2^-(\mathbb{C}P) + \frac{1}{2}(L.(K + 3L) + \chi(\mathbb{R}X) - \mu^-) + t_2 - \beta \\ &\quad + \frac{1}{2} \min(\tilde{b}_0(\mathbb{C}A''), \tilde{b}_0(\mathbb{C}A) + 2\beta, b_2(\mathbb{C}A) - \nu) \end{aligned}$$

$$(4-4) \quad \begin{aligned} \sigma_-(\bar{\phi}) + \sigma_0(\bar{\phi}) &\leq b_2^-(\mathbb{C}P) + \frac{1}{2}(L.(K + 3L) + \chi(\mathbb{R}X) - \mu^-) + 2t_2 + b_1(\mathbb{C}P) \\ &\quad + (b_2(\mathbb{C}A') - \nu') + \max(0, 3\alpha_{\text{Im}}^{(0)} - 1) \end{aligned}$$

If $\mathbb{C}A$ is a QI^S -curve, then $\alpha_{\text{Im}}^{(0)} = \beta = 0$, $\mathbb{C}A = \mathbb{C}A' = \mathbb{C}A''$. $\mathfrak{p} = \frac{1}{2}\mu^+$ and Theorem 4.1.1 can be applied to estimate σ_\pm , σ_0 of the both $\phi_+ = \frac{1}{2}\bar{\phi}$ and ϕ_- . This simplifies the formulae (4-1)–(4-4) as follows

4.1.2. Corollary. *Assume that $\mathbb{C}A$ is a QI^S -curve and $b_1(\mathbb{C}P; \mathbb{Z}/2) = 0$, then*

$$(4-5) \quad \sigma_+(\phi_\varepsilon) \leq b_2^+(\mathbb{C}P) + \frac{1}{2}L.(K + L) - \frac{1}{2}\mu^+ + \min(r - \nu - \sigma_0(\phi_\varepsilon), 0)$$

$$(4-6) \quad \begin{aligned} \sigma_-(\phi_\varepsilon) &\leq b_2^-(\mathbb{C}P) + \frac{1}{2}(L.(K + 3L) + \chi(\mathbb{R}X) - \mu^-) + \min(r - \nu - \sigma_0(\phi_\varepsilon), 0) \end{aligned}$$

where r is the number of irreducible components of $\mathbb{C}A$ and $\varepsilon \in \{+, -\}$.

Proof of Theorem 4.1.1. Following the scheme of the proof of Theorem 3.1.2, we need only to interpret

$$\begin{aligned}\sigma_{\pm}(\phi_{\varepsilon}) &\leq b_2^{\pm}(\overline{X}') \\ \sigma_{\pm}(\phi_{\varepsilon}) + \sigma_0(\phi_{\varepsilon}) &\leq b_2^{\pm}(\overline{X}') + b_2^0(\overline{X}') + \delta(\mathbb{C}X')\end{aligned}$$

modifying the left-hand side in accord with the identities of Lemma 4.1.3 and the estimates of $\delta(\mathbb{C}X') = \dim \ker(H_2(\mathbb{R}X) \rightarrow H_2(\overline{X}'))$ in Lemma 4.1.4. \square

4.1.3. Lemma. *Let $\gamma = b_3(\overline{X}) - b_1(\overline{X})$. Then*

(4-7)

$$b_2^0(\overline{X}') = b_2^0(\overline{X}^{\circ}) = \beta - \gamma$$

(4-8)

$$b_2^+(\overline{X}') = b_2^+(\overline{X}^{\circ}) = b_2^+(\mathbb{C}P) - b_1(\mathbb{C}P) + \frac{1}{2}L \cdot (K + L) - \mathfrak{p} + b_3(\overline{X})$$

(4-9)

$$\begin{aligned}b_2^-(\overline{X}') &= b_2^-(\overline{X}^{\circ}) = b_2^-(\mathbb{C}P) - b_1(\mathbb{C}P) + \frac{1}{2}(L \cdot (K + 3L) + \chi(\mathbb{R}X) - \mu^-) \\ &\quad + b_3(\overline{X}) - \beta\end{aligned}$$

4.1.4. Lemma. *Assume that $\mathbb{C}X \rightarrow \mathbb{C}P$ is a morphism of real surfaces being a double covering branched along a real reduced curve $\mathbb{C}A \subset \mathbb{C}P$, where $\mathbb{C}P$ is non-singular. Then*

(4-10)

$$b_1(\overline{X}) = \frac{1}{2}b_1(\mathbb{C}X) \leq b_1(\mathbb{C}P; \mathbb{Z}/2) + \frac{1}{2}\tilde{b}_0(A)$$

(4-11)

$$\begin{aligned}b_3(\overline{X}) &= \frac{1}{2}b_3(\mathbb{C}X) \leq b_1(\mathbb{C}P; \mathbb{Z}/2) + \frac{1}{2}\min(\tilde{b}_0(\mathbb{C}A''), \tilde{b}_0(\mathbb{C}A) + 2\beta, b_2(\mathbb{C}A) - \nu) \\ b_3(\overline{X}) &\leq b_1(\mathbb{C}P; \mathbb{Z}/2) + \tilde{b}_0(\mathbb{C}A')\end{aligned}$$

(4-12)

$$\delta(\mathbb{C}X') \leq 2b_1(\mathbb{C}P; \mathbb{Z}/2) - b_1(\overline{X}) + (b_2(\mathbb{C}A') - \nu') + \max(0, 3\alpha_{\text{Im}}^{(0)} - 1)$$

(4-13)

$$b_2^0(\overline{X}) + \delta(\mathbb{C}X') \leq 2b_1(\mathbb{C}P; \mathbb{Z}/2) - b_3(\overline{X}) + (b_2(\mathbb{C}A') - \nu') + \beta + \max(0, 3\alpha_{\text{Im}}^{(0)} - 1)$$

The proof of Lemma 4.1.4 is given in Appendix A.

Proof of Lemma 4.1.3. Like in the proof of Proposition 3.1.4, we have $b_2^{\pm}(\overline{X}') = b_2^{\pm}(\overline{X}^{\circ})$ and $b_2^0(\overline{X}') = b_2^0(\overline{X}^{\circ})$. Similarly, we obtain (4-1) together with the relations

$$\begin{aligned}(4-14) \quad b_2^+(\overline{X}^{\circ}) &= \frac{1}{2}(b_2^+(X^{\tau}) - 1) - \mathfrak{p} + b_3(\overline{X}) - b_1(\overline{X}) \\ b_2^-(\overline{X}^{\circ}) &= \frac{1}{2}(b_2^-(X^{\tau}) - 1 + \chi(X_{\mathbb{R}}) - \mu^-) - \beta + b_3(\overline{X}) - b_1(\overline{X})\end{aligned}$$

where $b_1(\overline{X}) = \frac{1}{2}b_1(\mathbb{C}X)$ by (4-10).

Comparing the Riemann-Hurwitz formula for the covering $\mathbb{C}X^\tau \rightarrow \mathbb{C}P$ with the formula for the signature of an involution applied to the covering transform, we obtain, like in the case $\mathbb{C}P = \mathbb{C}P^2$ considered in the introduction,

$$(4-15) \quad \frac{1}{2}(b_2^+(\mathbb{C}X^\tau) - b_1(\mathbb{C}X^\tau) + 1) = b_2^+(\mathbb{C}P) - b_1(\mathbb{C}P) + 1 - \frac{1}{8}(2\chi(\mathbb{C}A^\tau) + \langle \mathbb{C}A, \mathbb{C}A \rangle_{\mathbb{C}P})$$

$$(4-16) \quad \frac{1}{2}(b_2^-(\mathbb{C}X^\tau) - b_1(\mathbb{C}X^\tau) + 1) = b_2^-(\mathbb{C}P) - b_1(\mathbb{C}P) + 1 - \frac{1}{8}(2\chi(\mathbb{C}A^\tau) - \langle \mathbb{C}A, \mathbb{C}A \rangle_{\mathbb{C}P})$$

where, by the adjunction formula,

$$\begin{aligned} -\frac{1}{8}(2\chi(\mathbb{C}A^\tau) + \langle \mathbb{C}A, \mathbb{C}A \rangle_{\mathbb{C}P}) &= \frac{1}{2}L(K + L) \\ -\frac{1}{8}(2\chi(\mathbb{C}A^\tau) - \langle \mathbb{C}A, \mathbb{C}A \rangle_{\mathbb{C}P}) &= \frac{1}{2}L.(K + 3L) \end{aligned}$$

Combining (4-14) with (4-15) and (4-16), we obtain (4-8) and (4-9). \square

4.2. Generalized Arnold–Viro inequalities for surfaces. Consider a very good resolution $\text{res}: \mathbb{C}X^{\text{res}} \rightarrow \mathbb{C}X$, of a $\overline{\mathbb{Q}\mathbb{I}}$ -surface $\mathbb{C}X$ and put $\mathbb{C}U^{\text{res}} = \text{res}^{-1}(\mathbb{C}U)$, where $\mathbb{C}U$ is a regular neighborhood of $\text{Sing}(\mathbb{C}X)$. Let us call a point, $x \in \text{Sing}(\mathbb{C}X)$, $\mathbb{Z}/2$ -inessential if its link is a $\mathbb{Z}/2$ -homology sphere, and $\mathbb{Z}/2$ -essential otherwise. $\mathbb{Z}/2$ -essential imaginary singular points in $\mathbb{C}X$ split into conjugated pairs, whose number we denote by $\alpha_{\text{Im}}^{(2)}$. Consider $\mathbb{C}U_{\mathbb{R}} \subset \mathbb{C}U$ consisting of the connected components of $\mathbb{C}U$ around the real singularities of $\mathbb{C}X$ and denote by $d_{\mathbb{R}}^{(2)}$ the rank of $\mathbb{Z}/2$ -torsion of $H_1(\partial \mathbb{C}U_{\mathbb{R}})$. It is not difficult to check that $d_{\mathbb{R}}^{(2)}$ can be equivalently defined as the nullity of the (mod 2)-intersection-form in $\mathbb{C}U_{\mathbb{R}}^{\text{res}} = \text{res}^{-1}(\mathbb{C}U_{\mathbb{R}})$, or as the rank of $\text{Discr} \otimes \mathbb{Z}/2$, where Discr is the discriminant group of the lattice $H_2(\mathbb{C}U_{\mathbb{R}}^{\text{res}})$. Hence, $d_{\mathbb{R}}^{(2)}$ can be easily computed as soon as we know the resolution graph of $\mathbb{C}X^{\text{res}}$.

4.2.1. Theorem. *Assume that $\mathbb{C}X$ is a real $\overline{\mathbb{Q}\mathbb{I}}$ -surface with the partition form ϕ . Then*

$$(4-17) \quad \sigma_+(\phi) \leq p_g(\mathbb{C}X^{\text{res}})$$

$$(4-18) \quad \sigma_+(\phi) + \sigma_0(\phi) \leq \chi_a(\mathbb{C}X^{\text{res}}) + b_1(\mathbb{C}X^{\text{res}}; \mathbb{Z}/2) + \beta + \max(0, \alpha_{\text{Im}}^{(2)} - 1) + d_{\mathbb{R}}^{(2)}$$

$$(4-19) \quad \sigma_-(\phi) \leq \frac{1}{2}(b_2^-(\mathbb{C}X^{\text{res}}) - 1 + \chi(\mathbb{R}X) + \hat{\chi}(\mathbb{R}E)) - b_2(\overline{E})$$

$$\sigma_-(\phi) + \sigma_0(\phi) \leq \frac{1}{2}(b_2^-(\mathbb{C}X^{\text{res}}) + 1 + \chi(\mathbb{R}X) - \hat{\chi}(\mathbb{C}E)) + b_1(\mathbb{C}X^{\text{res}}; \mathbb{Z}/2)$$

$$(4-20) \quad -\frac{1}{2}b_1(\mathbb{C}X^{\text{res}}) + \max(0, \alpha_{\text{Im}}^{(2)} - 1) + d_{\mathbb{R}}^{(2)}$$

where $\hat{\chi}(Z) = \chi(Z) - b_0(Z)$ is the reduced Euler characteristic (in our particular case, $Z = \mathbb{C}E$ or $Z = \mathbb{R}E$).

Proof. We follow again the standard scheme of proving the Arnold-Viro-type inequalities, using the following lemmas

4.2.2. Lemma. *In the assumptions of Theorem 4.2.1 we have*

(4-21)

$$b_2^0(\overline{X}^\circ) = \beta - \gamma$$

(4-22)

$$b_2^+(\overline{X}^\circ) = b_2^+(\overline{X}^{\text{res}}) = p_g(\mathbb{C}X^{\text{res}})$$

(4-23)

$$b_2^-(\overline{X}^\circ) = b_2^-(\overline{X}^{\text{res}}) - b_2(\overline{U}^{\text{res}}) = \frac{1}{2}(b_2^-(\mathbb{C}X^{\text{res}}) - 1 + \chi(\mathbb{R}X) + \hat{\chi}(\mathbb{R}E)) - b_2(\overline{E})$$

4.2.3. Lemma. *For any real surface $\mathbb{C}X$ with normal singularities we have*

$$(4-24) \quad \delta(\mathbb{C}X') \leq b_1(\mathbb{C}X^{\text{res}}; \mathbb{Z}/2) - \frac{1}{2}b_1(\mathbb{C}X) + \max(1, \alpha_{\text{Im}}^{(2)}) + d_{\mathbb{R}}^{(2)}$$

(4-25)

$$b_2^0(\overline{X}^\circ) + \delta(\mathbb{C}X') \leq b_1(\mathbb{C}X^{\text{res}}; \mathbb{Z}/2) - \frac{1}{2}b_1(\mathbb{C}X^{\text{res}}) + \beta + \max(1, \alpha_{\text{Im}}^{(2)}) + d_{\mathbb{R}}^{(2)}$$

So, (4-17) and (4-19) are immediate corollaries of (4-22) and (4-23), whereas (4-18) follows from (4-17) and (4-25), since $\chi_a(\mathbb{C}X^{\text{res}}) = p_g(\mathbb{C}X^{\text{res}}) - \frac{1}{2}b_1(\mathbb{C}X^{\text{res}}) + 1$. (4-20) follows from (4-19) and (4-25), since $\frac{1}{2}\hat{\chi}(\mathbb{R}E) - b_2(\overline{E}) + \beta = -\frac{1}{2}\hat{\chi}(\mathbb{C}E)$. \square

The proof of Lemma 4.2.3 is given in Appendix A.

Proof of Lemma 4.2.2. The relation (4-21) is a version of (3-9) (a minor difference in the setting is not essential for the proof). The following proof of (4-22) and (4-23) is also similar to the proof of Theorem 3.3.1 (with $\mathbb{C}X^\tau$ being replaced by $\mathbb{C}X^{\text{res}}$ because the singularities of $\mathbb{C}X$ may be not ICIS).

Using the duality and the excision theorem, we obtain $b_2(\overline{X}^\circ) = b_2(\overline{X}^{\text{res}}, \overline{U}^{\text{res}})$, which together with the exact sequence of $(\overline{X}^{\text{res}}, \overline{U}^{\text{res}})$ gives

$$b_2(\overline{X}^\circ) = b_2(\overline{X}^{\text{res}}) - b_2(\overline{U}^{\text{res}}) + b_1(\overline{U}^{\text{res}}) - b_1(\overline{X}^{\text{res}}) + b_1(\overline{X}^{\text{res}}, \overline{U}^{\text{res}}) - \tilde{b}_0(\overline{U}^{\text{res}})$$

because the inclusion homomorphism $H_2(\overline{U}^{\text{res}}) \rightarrow H_2(\overline{X}^{\text{res}})$ is monomorphic (since $\overline{U}^{\text{res}}$ has non-degenerated intersection form). Using that $b_1(\overline{U}^{\text{res}}) = b_1(\overline{E}) = \beta$, $b_1(\overline{X}^{\text{res}}, \overline{U}^{\text{res}}) - \tilde{b}_0(\overline{U}^{\text{res}}) = b_1(\overline{X})$, and $b_1(\overline{X}^{\text{res}}) = b_3(\overline{X}^{\text{res}}) = b_3(\overline{X})$ (note that the relativization homomorphism $H_3(\overline{X}^{\text{res}}) \rightarrow H_3(\overline{X}^{\text{res}}, \overline{U}^{\text{res}})$ is isomorphism), we obtain

$$(4-26) \quad \begin{aligned} b_2(\overline{X}^\circ) &= b_2(\overline{X}^{\text{res}}) - b_2(\overline{U}^{\text{res}}) + \beta - \gamma \\ b_2^+(\overline{X}^\circ) + b_2^-(\overline{X}^\circ) &= b_2(\overline{X}^{\text{res}}) - b_2(\overline{U}^{\text{res}}) \end{aligned}$$

Furthermore, $\sigma(\overline{X}^{\text{res}}) = \sigma(\overline{X}^\circ) + \sigma(\overline{U}^{\text{res}}) = \sigma(\overline{X}^\circ) - b_2(\overline{U}^{\text{res}})$, since $b_2(\overline{U}^{\text{res}}) = b_2^-(\overline{U}^{\text{res}})$. This yields

$$2b_2^+(\overline{X}^\circ) = b_2(\overline{X}^\circ) - b_2^0(\overline{X}^\circ) + \sigma(\overline{X}^{\text{res}}) + b_2(\overline{U}^{\text{res}}) = b_2(\overline{X}^{\text{res}}) + \sigma(\overline{X}^{\text{res}}) = 2b_2^+(\overline{X}^{\text{res}})$$

and (4-22) follows, since $b_2^+(\overline{X}^{\text{res}}) = \frac{1}{2}(b_2^+(\mathbb{C}X^{\text{res}}) - 1) = p_g(\mathbb{C}X^{\text{res}})$ (cf. (1-4)).

The first equality in (4-23) is obtained from (4-22) and (4-26), whereas the second one uses (1-5) (or (3-3)). \square

4.3. Sharpness of the generalized Arnold-Viro inequalities. In this subsection we characterize the gaps between the left hand side and the right hand side in the generalized Arnold-Viro inequalities. Here I confine myself with the case of $\mathbb{Q}\mathbb{I}^S$ -curves and assume in addition $b_1(\mathbb{C}P) = 0$ and $b_0(\mathbb{C}A) = 0$ (this simplify the formulations, but for the arguments it is not very essential). Note that these conditions imply that $b_1(X) = 0$ for $X \rightarrow \mathbb{C}P$ being as above. Let us put

$$\begin{aligned}\Delta_\varepsilon^+ &= b_2^+(\mathbb{C}P) + \frac{1}{2}(L.(K + L) - \mu^+) - \sigma_+(\phi_\varepsilon) \\ \Delta_\varepsilon^- &= b_2^-(\mathbb{C}P) + \frac{1}{2}(L(K + 3L) + \chi(\mathbb{R}X^\varepsilon) - \mu^-) - \sigma_-(\phi_\varepsilon) \\ \Delta_\varepsilon^0 &= (r - \nu) - \sigma_0(\phi_\varepsilon)\end{aligned}$$

where Δ_ε^0 can be negative, although Δ_ε^\pm and $\Delta_\varepsilon^\pm + \Delta_\varepsilon^0$ cannot, by Corollary 4.1.2.

Let $g(\mathbb{C}A)$ denote the geometric genus of the curve $\mathbb{C}A$, and $g_a(\mathbb{C}A)$ the arithmetic genus. Let $n = b_0(\mathbb{R}P_\varepsilon - \mathbb{R}A)$ be the number of the partition components, W_i , and n_ω the number of those, which are not involved in the partition forms ϕ_\pm , i.e., for which $\omega|_{W_i} \neq 0$. Denote by 2br_{im} the total number of the imaginary branches at the singular points of $\mathbb{C}A$ (the number of branches at the imaginary singularities plus the number of the imaginary branches at the real singularities). Let α_{Im} denote the number of pairs of the imaginary singularities of $\mathbb{C}A$ and α_+ the number of positive dot-singularities

4.3.1. Proposition. *In the above assumptions on the surface $\mathbb{C}P$ and curve $\mathbb{C}A$, we have*

$$\Delta_\varepsilon = b_2(\mathbb{C}P) - \nu + g(\mathbb{C}A) + \text{br}_{\text{im}} - (\alpha_{\text{Im}} + \alpha_+) - b_1(\text{Int } \mathbb{R}P_\varepsilon) + n_\omega + b_2(\mathbb{R}P_\varepsilon)$$

4.3.2. Corollary. *The generalized Arnold-Viro inequalities are equalities if $\mathbb{C}P = \mathbb{C}P^2$, $\mathbb{C}A$ splits into rational irreducible components one of which has odd degree and all the singularities of $\mathbb{C}A$ are real and have only real branches.*

4.3.3. Corollary. *Under the assumption of Corollary 4.3.2 the radical of the partition form ϕ_ε has rank $r - 1$. In particular, the matrix of ϕ_ε is singular, since $r \geq 2$.*

For a singularity at $x \in \text{Sing}(\mathbb{C}A)$ the Milnor formula [Mi, Theorem 10.5] gives a relation $\delta_x = \frac{1}{2}(\mu_x + r_x - 1)$, where μ_x is the Milnor number, r_x is the number of branches of $\mathbb{C}A$ at x and δ_x is the maximal number of nodes which can appear after a deformation of this singularity. Denote by ρ_x the number of *real* branches at $x \in \text{Sing}(\mathbb{R}A)$. We have the following relations for an irreducible curve $\mathbb{C}A$ in a non-singular surface $\mathbb{C}P$.

$$(4-27) \quad 2 \sum_{x \in \text{Sing}(\mathbb{C}A)} \delta_x - \sum_{x \in \text{Sing}(\mathbb{C}A)} \mu_x = \sum_{x \in \text{Sing}(\mathbb{C}A)} (r_x - 1)$$

$$(4-28) \quad \sum_{x \in \text{Sing}(\mathbb{C}A)} (r_x - 1) - \sum_{x \in \text{Sing}(\mathbb{R}A)} (\rho_x - 1) = 2 \text{br}_{\text{im}} - 2\alpha_{\text{Im}}$$

$$(4-29) \quad \chi(\mathbb{R}A) = - \sum_{x \in \text{Sing}(\mathbb{R}A)} (\rho_x - 1)$$

$$(4-30) \quad g_a(\mathbb{C}A) + (r - 1) = g(\mathbb{C}A) + \sum_{x \in \text{Sing}(\mathbb{C}A)} \delta_x$$

(4-27) and (4-28) are obvious and (4-29) is a straightforward estimate of $\chi(\mathbb{R}A)$. (4-30) is the Plücker formula, see [Se, p.74] and [Mi, Property 10.5]. In turn, we note that existence of $A\Gamma$ -morsifications [AC,GZ] for real curve singularities (cf. §5) makes the proof of (4-30) elementary, reducing it to the trivial case of nodal curves, because $A\Gamma$ -morsifications obviously preserve r , g and $\sum_{x \in \text{Sing}(\mathbb{C}A)} \delta_x$.

Proof of Proposition 4.3.1. Adding the expressions defining Δ_ε^+ , Δ_ε^- and Δ_ε^0 , we obtain

$$\Delta = b_2(\mathbb{C}P) + L.(K + 2L) + \frac{1}{2}\chi(\mathbb{R}X^\varepsilon) - \frac{1}{2}\mu + (r - \nu) - (n - n_\omega)$$

The adjunction formula yields $L.(K + 2L) = g_a(\mathbb{C}A) - 1$, whereas (4-27)—(4-30) imply that

$$g_a(\mathbb{C}A) + (r - 1) - \frac{1}{2}\mu + \frac{1}{2}\chi(\mathbb{R}A) = g(\mathbb{C}A) + \text{br}_{\text{im}} - \alpha_{\text{Im}}$$

Furthermore, obviously $\frac{1}{2}\chi(\mathbb{R}X^\varepsilon) = \frac{1}{2}\chi(\mathbb{R}A) + \chi_c(\mathbb{R}P_\varepsilon - \mathbb{R}A)$ and $\chi_c(\mathbb{R}P_\varepsilon - \mathbb{R}A) = \chi_c(\text{Int } \mathbb{R}P_\varepsilon) - \alpha_+ = \chi(\text{Int } \mathbb{R}P_\varepsilon) - \alpha_+$, which yields finally

$$\begin{aligned} \Delta_\varepsilon &= b_2(\mathbb{C}P) + (g_a(\mathbb{C}A) - 1 + \frac{1}{2}\chi(\mathbb{R}A) - \frac{1}{2}\mu + r) - \nu - n + n_\omega + \chi(\text{Int } \mathbb{R}P_\varepsilon) - \alpha_+ \\ &= b_2(\mathbb{C}P) - \nu + g(\mathbb{C}A) + \text{br}_{\text{im}} - (\alpha_{\text{Im}} + \alpha_+) - b_1(\text{Int } \mathbb{R}P_\varepsilon) + n_\omega + b_2(\mathbb{R}P_\varepsilon) \end{aligned}$$

□

4.4. Generalized Petrovskii inequalities. If we omit in the Arnold-Viro-type inequalities any assumptions on the singularities of an irreducible even curve $\mathbb{C}A$ in a non-singular surface $\mathbb{C}P$, then some weaker estimates, $b_2^\pm(\bar{X}^\circ) \geq 0$, still can be used. One of them, $b_2^+(\bar{X}^\circ) \geq 0$, gives an estimate for \mathfrak{p}

$$b_2^+(\mathbb{C}P) + \frac{1}{2}L.(K + L) + t_2 + \frac{1}{2} \min(\tilde{b}_0(\mathbb{C}A''), \tilde{b}_0(\mathbb{C}A) + 2\beta, b_2(\mathbb{C}A) - \nu) \geq \mathfrak{p}$$

(the author cannot say much about its use and novelty). The other one, however, being the Petrovskii-type inequality, contains a somewhat non-trivial information about $\chi(\mathbb{R}X^\varepsilon)$

$$(4-31) \quad b_2^-(\mathbb{C}P) + \frac{1}{2}L.(K + 3L) - \frac{1}{2}\mu^- + t_2 - \beta + \frac{1}{2} \min(\tilde{b}_0(\mathbb{C}A''), \tilde{b}_0(\mathbb{C}A) + 2\beta, b_2(\mathbb{C}A) - \nu) \geq -\frac{1}{2}\chi(\mathbb{R}X^\varepsilon),$$

Applying (4-31) both for $\mathbb{R}X^+$ and $\mathbb{R}X^-$, and observing that $\chi(\mathbb{R}X^\varepsilon) = \chi(\mathbb{R}P_\varepsilon) + \chi(\mathbb{R}P) - \chi(\mathbb{R}P_{-\varepsilon})$, we obtain

$$|\chi(\mathbb{R}P_+) - \chi(\mathbb{R}P_-)| \leq 2b_2^-(\mathbb{C}P) + \chi(\mathbb{R}P) + L.(K + 3L) - \mu^- + 2t_2 \\ + \min(\tilde{b}_0(\mathbb{C}A'') - 2\beta, \tilde{b}_0(\mathbb{C}A), b_2(\mathbb{C}A) - \nu - 2\beta)$$

In the classical case $\mathbb{C}P = \mathbb{C}P^2$, it gives
(4-32)

$$|\chi(\mathbb{R}P_+) - \chi(\mathbb{R}P_-)| \leq 3k(k-1) + 1 - \mu^- + \min(0, \tilde{b}_0(\mathbb{C}A'') - 2\beta, b_2(\mathbb{C}A) - \nu - 2\beta)$$

which is a refinement of the generalized Petrovskii inequality stated by O. Ya. Viro [V1].

Remarks.

- (1) Another version of the generalized Petrovskii inequalities for real surfaces can be obtained if we use the estimate (4-19):

$$h^{1,1}(\mathbb{C}X^{\text{res}}) - 2 - 2b_2(\bar{E}) + \hat{\chi}(\mathbb{R}E) \geq -\chi(\mathbb{R}X).$$

- (2) Recall that V.M. Kharlamov announced [Kh1] a different sort of the generalized Petrovskii inequalities. The relation of Kharlamov's generalization to the generalization of Viro and to the formula (4-32) seems to be an open question yet.

§5. COMPUTATION OF THE FORMS q_\pm

5.1. The local partition forms. Assume that $f: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ is an isolated singularity defined by a real polynomial $f(x, y)$ with the zero locus $\mathbb{C}A \subset \mathbb{C}^2$. By $A\Gamma$ -morsification of f (“ $A\Gamma$ ” refers to A’Campo and Gusein-Zade [AC,GZ]) we mean a small real deformation, f^τ , of f having μ non-degenerate *real* critical points in an ε -ball, $\mathbb{R}B \subset \mathbb{C}^2$, $0 \ll |\tau| \ll \varepsilon \ll 1$, around zero (μ is the Milnor number) and the maximal possible number, $\frac{1}{2}(\mu + \rho - 1)$, of saddle points for $f^\tau|_{\mathbb{R}B}$, which all lie on the same level curve $\mathbb{C}A^\tau = \{f^\tau = 0\}$ (ρ here, like in §4, is the number of the branches of $\mathbb{R}A$ at 0). Along with such objects as $\mathbb{R}B_\pm$, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, which were assigned to a singularity f in 2.4, we consider their deformations, $\mathbb{R}B_\pm^\tau = \mathbb{R}B \cap \{\pm f^\tau(x, y) \geq 0\}$, and $\mathcal{H}_\pm^\tau = H^0(\mathbb{R}B_\pm^\tau \setminus \mathbb{R}A^\tau)$, $\mathcal{H}^\tau = \mathcal{H}_+^\tau \oplus \mathcal{H}_-^\tau$. Denote by W_i^τ the closures of the connected components of $\mathbb{R}B^\tau \setminus \mathbb{R}A^\tau$, so that the negative indices, $i = -l, \dots, -1$ are used for those of the components which lie in the interior of $\mathbb{R}B$, and positive, $i = 1, \dots, 2\rho$, for those which have common points with $\mathbb{R}S = \partial(\mathbb{R}B)$, if $\rho \geq 1$. In the case of a dot singularity (i.e., $\rho = 0$), there is only one component bounded by $\mathbb{R}S = \partial\mathbb{R}B$, which we denote by W_0^τ .

The regions W_i^τ associated to f^τ will be called *$A\Gamma$ -regions* and the diagram $(\mathbb{R}B, \mathbb{R}B \cap \mathbb{R}A^\tau)$ characterizing the mutual adjacencies of W_i^τ , will be called *$A\Gamma$ -diagram*.

Denote by W_i the closures of the connected components of $\mathbb{R}B \setminus \mathbb{R}A$ which are deformed into W_i^τ , $i \geq 0$. Like before, we put $\text{sign}(W_i^\tau) \in \{+, -\}$ for the sign of f^τ inside W_i^τ , and $\text{sign}(W_i) \in \{+, -\}$ for the sign of f inside W_i . We denote

by $w_i^\tau \in \mathcal{H}^\tau$ the generators representing W_i^τ and by $w_i \in H^0(\mathbb{R}B \setminus \mathbb{R}A) \cong \mathcal{H}$ the generators representing W_i . Let us define a quadratic form, $q^\tau: \mathcal{H}^\tau \rightarrow \mathbb{Q}$, putting

$$\begin{aligned} q^\tau(w_i^\tau, w_j^\tau) &= 0 && \text{if } \text{sign}(W_i^\tau) \neq \text{sign}(W_j^\tau) \\ q^\tau(w_i^\tau, w_j^\tau) &= \frac{1}{2} \text{ord}(W_i^\tau \cap W_j^\tau); && \text{if } \text{sign}(W_i^\tau) = \text{sign}(W_j^\tau), i \neq j \\ q^\tau(w_i^\tau, w_i^\tau) &= \frac{1}{2} \text{ord}(W_i^\tau \cap \text{Cl}(\mathbb{R}B_\varepsilon^\tau - W_i^\tau)) - 2\chi(W_i^\tau \setminus \mathbb{R}S) && \text{if } -l \leq i \leq 2\rho \end{aligned}$$

where ord denotes the number of points. Let \mathcal{E} denote the subspace of \mathcal{H}^τ generated by w_i^τ , $i \in \{-1, \dots, -l\}$. If the restriction of q^τ to \mathcal{E} is non-degenerated, then we obtain a direct sum decomposition, $\mathcal{H}^\tau = \mathcal{E} \oplus \mathcal{E}^\perp$, where \mathcal{E}^\perp denotes the orthogonal complement to \mathcal{E} with respect to the quadratic form q^τ . Denote by \hat{w}_i^τ , the component of w_i^τ in \mathcal{E}^\perp with respect to this direct sum decomposition, where $i = 1, \dots, 2\rho$ for $\rho \geq 1$, and $i = 0$ for $\rho = 0$.

Consider $\mathcal{E}_\pm = \mathcal{E} \cap \mathcal{H}_\pm^\tau$ and $\mathcal{E}_\pm^\perp = \mathcal{E}^\perp \cap \mathcal{H}_\pm^\tau$. It is obvious that $\mathcal{E}^\perp = \mathcal{E}_+^\perp \oplus \mathcal{E}_-^\perp$ and that \mathcal{E}_\pm^\perp is the orthogonal complement to \mathcal{E}_\pm with respect to the form $q_\pm^\tau = q^\tau|_{\mathcal{H}_\pm^\tau}$, provided the latter is non-degenerated.

5.1.1. Theorem. *Assume that $f: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ is an isolated real singularity and f^τ its $A\Gamma$ -morsification. Then*

- (1) *f is an $\overline{\mathbb{Q}\Gamma}^S$ -singularity if and only if the restriction of q^τ to \mathcal{E}_+ is non-degenerated.*
- (2) *Assume that f is an $\overline{\mathbb{Q}\Gamma}^S$ -singularity. Then $\bar{q}(w_i, w_j) = 2q_+^\tau(\hat{w}_i^\tau, \hat{w}_j^\tau)$, for any $A\Gamma$ -regions, W_i, W_j , if $0 \leq i, j \leq 2\rho$, $\text{sign}(W_i) = \text{sign}(W_j) = +$.*

Replacing f by $-f$, we obtain a version of Theorem 5.1.1 for q_-^τ and \mathcal{E}_- . Combined together, these two versions imply

5.1.2. Theorem. *Assume that f^τ is like in Theorem 5.1.1. Then*

- (1) *f is an $\mathbb{Q}\Gamma^S$ -singularity if and only if the restriction of q^τ to \mathcal{E} is non-degenerated.*
- (2) *Assume that f is an $\mathbb{Q}\Gamma^S$ -singularity. Then $q(w_i, w_j) = q^\tau(\hat{w}_i^\tau, \hat{w}_j^\tau)$, for any $A\Gamma$ -regions, W_i, W_j , $0 \leq i, j \leq 2\rho$.*

Note the above theorems reduce the problem of calculating the forms q , q_\pm to an elementary combinatorial analysis of $A\Gamma$ -diagrams (whose construction is known due to [AC,GZ]) and some trivial linear algebra (completion the squares of w_i^τ , $-l \leq i \leq -1$, in the form q^τ).

5.2. Contraction of 2-dimensional polyhedra in rational homology manifolds. Assume that Z is a compact oriented \mathbb{Q} -homology 4-manifold, for simplicity, a polyhedron (for our purpose, it suffices to consider only Whitney stratified pseudo-manifolds, which are known to carry a polyhedral structure), and $K \subset Z - \partial Z$ a 2-dimensional sub-polyhedron. Assume that $H_1(K; \mathbb{Q}) = 0$, the inclusion homomorphism, $\text{in}: H_2(K; \mathbb{Q}) \rightarrow H_2(Z; \mathbb{Q})$, is monomorphic and the restriction of the intersection form in Z to $\mathcal{E} = \text{in}(H_2(K; \mathbb{Q}))$ is non-degenerated. Analyzing the long homology sequence of the pair (Z, K) , one easily obtains that the quotient space Z/K is also a \mathbb{Q} -homology manifold, whose intersection form is isomorphic to the restriction of the intersection form in Z to the orthogonal complement, \mathcal{E}^\perp , of \mathcal{E} in $H_2(Z; \mathbb{Q})$.

5.3. Proof of Theorems 5.1.1. Let us denote by $\mathbb{C}X$ and $\mathbb{C}X^\tau$ the affine surfaces defined in \mathbb{C}^3 by the equations $f(x, y) - z^2 = 0$ and $f^\tau(x, y) - z^2 = 0$, and by π, π^τ the projections of $\mathbb{C}X, \mathbb{C}X^\tau$ to the (x, y) -plane, \mathbb{C}^2 . Put $\mathbb{C}U = \pi^{-1}(\mathbb{C}B)$, $\mathbb{C}U^\tau = (\pi^\tau)^{-1}(\mathbb{C}B)$, like in 2.4, and let $\Gamma_i^\tau = \pi^{-1}(W_i^\tau)$, $i \in \{-l, \dots, 2\rho\}$. Attaching 2-handles to \overline{U}^τ along the link $L = \partial \mathbb{R}U^\tau$, with the canonical framing, one obtains a 4-manifold, which we denote by \overline{U}_L^τ . Similarly attaching 2-handles to \overline{U} , we obtain a \mathbb{Q} -homology 4-manifold, \overline{U}_L . In the case $\rho \geq 1$, we denote by $\hat{\Gamma}_i^\tau \subset \overline{U}_L^\tau$, $i = 1, \dots, 2\rho$ the union of Γ_i^τ with the core of the corresponding handle. In the case of a positive dot singularity, let $\hat{\Gamma}_0^\tau$ be the union of Γ_0^τ with the cores of the both handles of \overline{U}_L^τ .

The fundamental classes, $[\Gamma_i^\tau]$, $i = -1, \dots, -l$, form a basis in $H_2(\mathbb{C}U^\tau)$. Similarly, those of these classes, with $\text{sign}(W_i^\tau) = +$, form a basis of $H_2(\overline{U}^\tau)$ (the both facts follow, for instance, from the description of the Milnor form in [AVG, section 1.4]). Together with $[\hat{\Gamma}_i^\tau]$, $i \geq 0$, $\text{sign}(W_i) = +$, the latter classes form a basis of $H_2(\overline{U}_L^\tau)$. Since $\mathbb{C}U^\tau$ is homotopy equivalent to a wedge of 2-spheres, as was mentioned in 3.3, the set $R = \cup_{-l \leq i \leq -1} \Gamma_i^\tau$ is a spine (a regular deformational retract) of $\mathbb{C}U^\tau$, whereas $R_+ = R/\text{conj}$ is a spine of \overline{U}^τ . This implies that $\mathbb{C}U \cong \mathbb{C}U^\tau/R$ and $\overline{U} \cong \overline{U}^\tau/R_+$.

The homology sequence of the pair $(\overline{U}^\tau, \partial \overline{U}^\tau)$, implies that the intersection form on \overline{U}^τ is non-degenerated if and only if $\partial \overline{U}^\tau$ is a \mathbb{Q} -homology sphere. On the other hand, it is known that the intersection form in $\mathbb{C}U^\tau$ is described by the restriction $q^\tau|_{\mathcal{E}}$ (see [AVG, 1.4]). Together with Proposition 2.1.1, this implies that the intersection form in \overline{U}^τ is described by the restriction of $\bar{q}^\tau = 2q^\tau$ to \mathcal{E}_+ and proves the first part of the Theorem.

Similar calculations show that the formulae for the intersection form in $H_2(\overline{U}_L^\tau) \cong \mathcal{H}_+^\tau$ are almost the same as for $2q_+^\tau$. The distinction arises only for the self-intersections, $\langle \hat{\Gamma}_i^\tau, \hat{\Gamma}_i^\tau \rangle_{\overline{U}_L^\tau}$, $i = 1, \dots, 2\rho$, for $\rho \geq 1$, which are equal to $2q_+^\tau(w_i, w_i) - 2\chi(W_i^\tau \cap \mathbb{R}S)$. On the other hand, it follows from the definition of λ and q that

$$\bar{\lambda}([\partial \Gamma_i], [\partial \Gamma_j]) = \langle \hat{\Gamma}_i, \hat{\Gamma}_j \rangle_{\overline{U}_L} = 2q_+(w_i, w_j) - 2\chi(W_i \cap \mathbb{R}S)\delta_{ij}$$

where δ_{ij} is the Kronecker symbol. Since, obviously, $\chi(W_i \cap \mathbb{R}S) = \chi(W_i^\tau \cap \mathbb{R}S) = 1$, for $\rho \geq 1$, and $\chi(W_i \cap \mathbb{R}S) = 0$ for $\rho = 0$, we obtain the second part of Theorem 5.1.1 applying the observation in section 5.2 to $Z = \overline{U}_L^\tau$ and $K = R_+$. \square

Remarks.

- (1) Calculation of $\langle \hat{\Gamma}_i, \hat{\Gamma}_j \rangle_{\overline{U}_L}$ is quite elementary: we just combine the formula (4-1) with the Example in section 2.4.3 showing that the form \bar{q}_\pm for a cross-like node is described by the matrix $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.
- (2) An elementary analysis of the links at the points $x \in \Gamma_i \cap \Gamma_j$ shows that \overline{U}^τ (and thus \overline{U}_L^τ) is a topological 4-manifold. After it is naturally smoothed, the intersection of Γ_i and Γ_j in \overline{U}^τ becomes transversal (see [F2]).

5.4. Local complex intersection forms for simple singularities. As an illustration, we present below the matrices M of the forms q_+ for the simple real surface singularities, which can be easily computed applying the algorithm described above to the $A\Gamma$ -diagrams (sketched in [AC], [GZ]) of real simple singularities.

In the matrices M , characterizing the singularities D_{2n+2}^- , D_{2n+3}^- and E_7 , the lesser diagonal entry corresponds to the most narrow region (the one bounded by the real branches which form the angle 0). The shape of the other singularities is symmetric, so we omit the correspondence between the regions and the entries of the matrices.

$$\begin{aligned}
f(x, y) &= \begin{cases} -x^{2n} + y^2; & A_{2n-1}^- & M = \begin{pmatrix} \frac{n}{2} & \frac{n}{2} \\ \frac{n}{2} & \frac{n}{2} \end{pmatrix} \\ x^{2n} - y^2; & A_{2n-1}^+ & M = \begin{pmatrix} \frac{2n-1}{2n} & \frac{1}{2n} \\ \frac{1}{2n} & \frac{2n-1}{2n} \end{pmatrix} \\ x^{2n} + y^2; & A_{2n-1}^{\circ-} & M = (2n-2) \\ -x^{2n} - y^2; & A_{2n-1}^{\circ+} & M = (0) \end{cases} \\
f(x, y) &= \begin{cases} \pm x^{2n+1} + y^2; & A_{2n}^- & M = (2n) \\ \pm x^{2n+1} - y^2; & A_{2n}^+ & M = (\frac{2n}{2n+1}) \end{cases} \\
f(x, y) &= \begin{cases} \pm x(x^{2n} - y^2); & D_{2n+2}^- & M = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{n+1}{2} & \frac{n}{2} \\ \frac{1}{2} & \frac{n}{2} & \frac{n+1}{2} \end{pmatrix} \\ \pm x(x^{2n} + y^2); & D_{2n+2}^+ & M = (2n) \end{cases} \\
f(x, y) &= \begin{cases} x(x^{2n+1} \pm y^2); & D_{2n+3}^- & M = \begin{pmatrix} 2n+1 & 1 \\ 1 & 1 \end{pmatrix} \\ -x(x^{2n+1} \pm y^2); & D_{2n+3}^+ & M = \begin{pmatrix} \frac{2n+3}{4} & \frac{2n+1}{4} \\ \frac{2n+1}{4} & \frac{2n+3}{4} \end{pmatrix} \end{cases} \\
f(x, y) &= \begin{cases} x^4 \pm y^3; & E_6^- & M = (6) \\ -x^4 \pm y^3; & E_6^+ & M = (2) \end{cases} \\
f(x, y) &= \pm y(x^3 \pm y^2); & E_7 & M = \begin{pmatrix} \frac{7}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} \end{pmatrix} \\
f(x, y) &= \pm x^5 \pm y^3; & E_8 & M = (8)
\end{aligned}$$

5.5. Some other methods and examples of computation of the forms

q_x . The forms q_x for real isolated normal surface singularities, can be described in terms of the resolution of singularities, which is, in principle, more general and sometimes more convenient then using $\text{A}\Gamma$ -morsifications. We sketch a method how to do it, although under certain restrictions on a singularity yet. Consider a compact conj-invariant regular cone-like neighborhood, $\mathbb{C}U$ of a real surface QI -point, x , and denote by D_i , $i = 1, \dots, \rho$, the closures of the connected components of $\mathbb{R}U - \{x\}$; D_i are the topological discs bounded by the components L_1, \dots, L_ρ of the real link $L = \partial(\mathbb{R}U)$. Assume that the exceptional divisor, $\mathbb{C}E$, of a very good resolution, $r: \mathbb{C}U^{\text{res}} \rightarrow \mathbb{C}U$, contains at least one real (that is conj-invariant) component. Then non-real components do not intersect $\mathbb{R}U^{\text{res}}$, since $\mathbb{C}E$ is connected and does not contain triple intersection points. Consider an intermediate resolution, $\text{res}': \mathbb{C}U' \rightarrow \mathbb{C}U$, where $\mathbb{C}U'$ is obtained from $\mathbb{C}U^{\text{res}}$ by contraction of all the non-real components of $\mathbb{C}E$. The following property is satisfied in numerous examples: there exists a disjoint set, $\mathbb{C}E^1, \dots, \mathbb{C}E^n \subset \mathbb{C}U'$, of the components of the exceptional divisor of res' with the 2-sided real parts $\mathbb{R}E^i \subset \mathbb{R}U'$, so that

$\mathbb{R}U' - \bigcup_{i=1}^n \mathbb{R}E^i$ split into ρ orientable connected components, and for the closures of these components, which we denote by F_i , $\text{res}'(F_i) = D_i$, $i = 1, \dots, \rho$. Choose arbitrarily orientations of D_i and the coherent orientations of F_i . Put $\varepsilon_k = 1$, $k = 1, \dots, n$, if the surfaces, F_i, F_j , adjacent to $\mathbb{R}E^k$ from its two sides (possibly $i = j$) induce the opposite orientations on $\mathbb{R}E^k$, otherwise put $\varepsilon_k = -1$. Denote by α_k the self-intersection of $\mathbb{C}E^k$ in $\mathbb{C}U'$. Note that α_k are determined by the resolution graph of $\mathbb{C}U^{\text{res}}$ and can be calculated by the well-known continued fractions algorithm [HKK] (this algorithm, in turn, is justified by the remark in 5.2).

Then for $\gamma_i = [L_i] \in H_1(L)$ we have

$$\begin{aligned}\lambda_x(\gamma_i, \gamma_j) &= -\frac{1}{4} \sum_{\mathbb{R}E^k \subset F_i \cap F_j} \varepsilon_k \alpha_k, \quad \text{for } i \neq j \\ \lambda_x(\gamma_i, \gamma_i) &= \frac{1}{4} \sum_{\mathbb{R}E^k \subset \partial F_i} \varepsilon_k \alpha_k - \chi(F_i)\end{aligned}$$

The proof is essentially reduced to a quite elementary analysis of the example 5.5.1 below.

Consider now the case of $\mathbb{C}E$ not containing conj-invariant components. Then $\mathbb{R}U^{\text{res}} = \mathbb{R}U$ is a non-singular surface, which is homeomorphic to a wedge of ρ discs, hence $\rho = 1$. Furthermore, $\mathbb{C}E - \{x\}$ splits into a pair of connected components. We denote by E', E'' the closures of these components and note that E' and E'' are permuted by conj and homeomorphic to \overline{E} . The definition of the resolution graph admits an obvious extension to the quotients, $\overline{E} \subset \overline{U}^{\text{res}}$; namely, such a graph (being a tree in our case) characterizes likewise the intersections of the components of \overline{E} in $\overline{U}^{\text{res}}$ (one can call it a *quotient resolution graph*). Such a graph almost coincides with a “half” of the usual resolution graph, i.e., a subgraph representing the components of E' (or, equally, of E''). The distinction arises only with the weight of one vertex corresponding to the component, E_x , of E' , which contains x (the weight in the quotient resolution graph is less by 1).

Let \hat{U} be obtained from $\overline{U}^{\text{res}}$ by contraction of all the components of \overline{E} except E_x ; then \hat{U} is a \mathbb{Q} -homology manifold containing E_x as a deformational retract. Using the continued fractions algorithm, we can determine the self-intersection $\alpha = \langle E_x, E_x \rangle_{\hat{U}}$ and obtain, applying the remark in 5.2, that

$$\lambda_x([L], [L]) = -\frac{4}{\alpha} - 1$$

where 4 appears as the square of $\langle E_x, \mathbb{R}U^{\text{res}} \rangle_{\hat{U}} = \pm 2$.

Example 5.5.1. Consider a rational $(-m)$ -curve $\mathbb{C}E$, $m > 0$, $m \in \mathbb{Q}$, in a QI-surface $\mathbb{C}X'$ such that the real part $\mathbb{R}E$ of $\mathbb{C}E$ (but not necessarily the whole $\mathbb{C}E$) is smooth. Then the linking form λ_x of the singularity which appears in $\mathbb{C}X = \mathbb{C}X'/\mathbb{C}E$ after contraction of $\mathbb{C}E$, is described by the matrix $(-m)$, if $\mathbb{R}E$ is one-sided in $\mathbb{R}X$, and by the matrix $\begin{pmatrix} -\frac{m}{4} & \frac{m}{4} \\ \frac{m}{4} & -\frac{m}{4} \end{pmatrix}$ if it is two-sided.

We can apply this method to calculate, for instance, the form q_x for the singularity A_{2n-1}^+ , since the latter appears after contraction of a rational curve $\mathbb{C}E$ with a pair of complex-conjugated imaginary singularities of the type A_{n-1} and with $\langle \mathbb{C}E, \mathbb{C}E \rangle_{\mathbb{C}X'} = -\frac{2}{n}$. This gives a matrix $\begin{pmatrix} -\frac{1}{2n} & \frac{1}{2n} \\ \frac{1}{2n} & -\frac{1}{2n} \end{pmatrix}$ of the form λ_x .

Example 5.5.2. We call a real surface singularity *quasi-cuspidal* if its real link $\mathbb{R}M$ has one component (an obvious example is the suspension over an unibranch curve singularity). If, for a quasi-cuspidal singularity, $\mathbb{R}U^{\text{res}}$ is orientable, then, putting $\mathbb{C}U' = \mathbb{C}U^{\text{res}}$, $F = \mathbb{R}U^{\text{res}}$, and applying the algorithm described in 5.4, we obtain $\lambda([L], [L]) = -\chi(F) = 2g - 1$, where g is the genus of F . If we are given an unibranch curve singularity, so that the suspension surface singularity has orientable real part $\mathbb{R}U^{\text{res}}$, then the residue form is determined by the value $q_+(v) = 2g$, where $v \in \mathcal{H}_+ \cong \mathbb{Z}$ is a generator.

Note that $\mathbb{R}U^{\text{res}}$ is orientable if the real components of $\mathbb{C}E$ have even self-intersections. The Lefschetz fixed point formula for the involution conj implies moreover that $\chi(\mathbb{R}U^{\text{res}}) = 1 - m$, where $m = 2g$ is the number of real components of $\mathbb{C}E$. For instance, for quasi-cuspidal singularities A_{2n}^- and D_{2n+2}^+ , the number of real components is equal to $2n$, for E_6^+ and E_8 , this number is 2 and 8 respectively, which determine their forms q_+ (cf. the table in 5.4).

Another example is the singularity $f: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$, $f(x, y) = x^{2n} \pm y^{2n-1}$, where n is even. In this case $\mathbb{C}U^{\text{res}}$ is spin, and thus $\mathbb{R}U^{\text{res}}$ is orientable. Furthermore, it is not difficult to determine the number, $m = 4n - 2$, of the real components of $\mathbb{C}E$, which implies that $q_+(v, v) = (4n - 2)$. It contrasts to the case of odd n , in which $\mathbb{C}U^{\text{res}}$ is not Spin and $q_+ = 0$ (see the following example).

Example 5.5.3. For the singularity at $0 \in \mathbb{C}^2$ defined by $f(x, y) = x^{2n} \pm y^{2n-1}$, where n is odd, the form q_+ vanishes. To see it, we consider the double plane, $\mathbb{C}X^- \rightarrow \mathbb{CP}^2$, branched along the projective closure, $\mathbb{C}A$, of the curve $\{f = 0\}$ defined by the equation $x^{2n} \pm y^{2n-1}z - w^2 = 0$. The quotient $\overline{X^-}$ is a double covering over $\mathbb{CP}^2/\text{conj} \cong S^4$ branched along $\mathfrak{A}_+ = \mathbb{RP}_+^2 \cup \overline{A}$ (cf. [Ar]), where \overline{A} is a 2-disc, because $\mathbb{C}A$ is a rational curve. Since \mathbb{RP}_+^2 is also a 2-disc, we have $\mathfrak{A}_+ \cong S^2$ and $\overline{X^-}$ is a homotopy sphere, because it is simply connected, as $\mathbb{C}X$ is (actually, one can show that $\overline{X^-}$ is diffeomorphic to S^4). This implies that $\langle \mathbb{R}X, \mathbb{R}X \rangle_{\overline{X^-}} = 0$ and thus $q_+ = 0$, because $\mathbb{R}X$ is a torus with a unique singularity.

Example 5.5.4. Assume that $p, q \geq 1$, $p + q = 2n$, $(p, 2n) = 1$ and n is odd. Then the singularities defined at 0 by $f(x, y) = x^{2n} - y^p$ and $g(x, y) = x^{2n} - y^q$, have the opposite forms, i.e., $q_f(v_f, v_f) = -q_g(v_g, v_g)$, where $v_f \in \mathcal{H}_{f+} \cong \mathbb{Z}$, $v_g \in \mathcal{H}_{g+} \cong \mathbb{Z}$ are generators.

To prove it, we apply the same arguments as in the previous example to the curve $\mathbb{C}A = \{x^{2n} - y^p z^{2n-p} = 0\} \subset \mathbb{CP}^2$.

§6. SOME EXAMPLES AND APPLICATIONS

6.1. Arrangements of hyperplanes. Consider a hypersurface $\mathbb{C}A \subset \mathbb{CP}^d$ which splits into $m = 2k$ real hyperplanes in generic position; then $\mathbb{C}A$ is a $\mathbb{Q}\mathbb{I}^S$ -hypersurface (see Example 2.4.3 in section 2.4). Let W_1, \dots, W_m denote the partition regions (i.e., the polyhedra bounded by the hyperplanes) of $\mathbb{C}A$. If $d - k$ is even, then we can choose as Ω (cf. section 2.5) one of the hyperplanes; if $d - k$ is odd, then we put $\Omega = \emptyset$.

We assume first that $\Gamma = W_i \cap W_j$ is a *connected* polyhedron (which is always the case if m is sufficiently large). Let $f_\Gamma(t)$ be the face-counting polynomial defined as

$$f_\Gamma(t) = \sum_{0 \leq k \leq s} f_k t^k$$

where $s = \dim \Gamma$ and f_k is the number of k -faces of Γ . If $\Gamma \subset \Omega$ and Ω locally (near Γ) separates W_i and W_j , then we put $\varepsilon(W_i, W_j|\Omega) = -1$, otherwise, we put $\varepsilon(W_i, W_j|\Omega) = 1$. Split the regions W_i into positive and negative assigning a sign, $\text{sign}(W_i) \in \{+, -\}$, so that W_i and W_j have no common $(d-1)$ face if $\text{sign}(W_i) = \text{sign}(W_j)$. Put $\text{sign}(W_i, W_j) = \frac{1}{2}(\text{sign}(W_i) - \text{sign}(W_j))$.

The formulae (2-9) easily imply the following result

6.1.1. Theorem. *Assume that $\Gamma = W_i \cap W_j$ is a connected polyhedron of dimension s . Then*

$$\phi(w_i, w_j|\Omega) = \varepsilon(W_i, W_j|\Omega)(-1)^{\frac{1}{2}(d^2-s+\text{sign}(W_i, W_j))} 2^{1-d} f_\Gamma(-2)$$

for any $i, j \in \{1, \dots, m\}$. \square

Recall that one can associate certain real quasi-smooth toric variety, $\mathbb{C}T_\Gamma$, to a polyhedron Γ . The algebraic structure of $\mathbb{C}T_\Gamma$ depends on the geometry of Γ (which actually must have rational vertices). However, the Poincare polynomial, $P_{\mathbb{C}T_\Gamma}(t)$ of $\mathbb{C}T_\Gamma$, is determined by the combinatorial type of Γ , namely, $P_{\mathbb{C}T_\Gamma}(t) = f_\Gamma(t^2 - 1)$ (see [Da]). In particular, this implies that

$$f_\Gamma(-2) = P_{\mathbb{C}T_\Gamma}(i) = \sigma(\mathbb{C}T_\Gamma) = -\chi(\mathbb{R}T_\Gamma)$$

Example: If $\Gamma = W_i \cap W_j$ is a simplex of dimension s , then $P_{\mathbb{C}T_\Gamma} = P_{\mathbb{C}P^s}$, so $|\phi(W_i, W_j|\Omega)| = (\frac{1}{2})^{d-1}$ if s is even and $\phi(W_i, W_j|\Omega) = 0$ if odd. In the case of a prism, $W_i \cap W_j \cong \Delta \times [0, 1]$, with any convex polyhedron Δ as the base, we have $\phi(W_i, W_j) = 0$. More generally, one can use the obvious relation $f_{\Gamma_1 \times \Gamma_2} = f_{\Gamma_1} \times f_{\Gamma_2}$ for evaluation of $\phi(W_i, W_j)$ if $W_i \cap W_j \cong \Gamma_1 \times \Gamma_2$.

If $W_i \cap W_j = \Gamma_1 \cup \dots \cup \Gamma_r$ is a union of several connected polyhedra, then $\phi(w_i, w_j|\Omega)$ is the sum of the expressions given by Theorem 6.1.1, calculated for each polyhedron Γ_i .

6.1.2. Theorem. *Assume that $\mathbb{C}A \subset \mathbb{C}P^d$ is as above, an arrangement of hyperplanes, $d = 2n$ and $\pi: X \rightarrow \mathbb{C}P^d$ the double covering branched along $\mathbb{C}A$. Denote by $\mathbb{C}X^\varepsilon$, $\varepsilon = +$ or $\varepsilon = -$, the complex variety X endowed with one of the two real structures (defined by the complex conjugations) lifted from $\mathbb{C}P^d$, and put $\mathbb{R}P_\varepsilon = \pi(\mathbb{R}X^\varepsilon)$. Let $\phi_\varkappa: H^0(\mathbb{R}P_\varepsilon - \mathbb{R}A) \rightarrow \mathbb{Q}$ denote the components of the partition form, $\phi = \phi_+ \oplus \phi_-$. Then*

$$\begin{aligned} \sigma_{-\varkappa}(\phi_\varkappa) &= \frac{1}{2}(b_d^{-\varkappa}(X) - \kappa) \\ \sigma_\varkappa(\phi_\varkappa) &= \frac{1}{2}(b_d^\varkappa(X) + \kappa + \chi(\mathbb{R}X^\varepsilon)) - 1 \\ \sigma_0(\phi_\varkappa) &= \sum_{k=0}^{d-1} \binom{m-2}{k} \end{aligned}$$

where \varkappa and κ are like in subsection 3.1.

This result shows that the generalized Arnold-Viro inequalities are indeed equalities for arrangements of hyperplanes.

Proof. The Arnold-Viro inequalities (3-1)–(3-2) give

$$(6-1) \quad \begin{aligned} \sigma_{-\varkappa}(\phi_{\varkappa}) + \sigma_0(\phi_{\varkappa}) &\leq \frac{1}{2}(b_d^{-\varkappa}(X) - \kappa) + \delta(\mathbb{C}X^{-\varepsilon}) \\ \sigma_{\varkappa}(\phi_{\varkappa}) &\leq \frac{1}{2}(b_d^{\varkappa}(X) + \kappa + \chi(\mathbb{R}X^{\varepsilon})) - 1 \end{aligned}$$

and thus $\sigma_+(\phi_{\varkappa}) + \sigma_-(\phi_{\varkappa}) + \sigma_0(\phi_{\varkappa}) \leq \frac{1}{2}(b_d(X) + \chi(\mathbb{R}X^{\varepsilon})) - 1 + \delta(\mathbb{C}X^{\varepsilon})$

Combining together these inequalities for $\varepsilon = +$ and $\varepsilon = -$, we obtain

$$(6-2) \quad N_d^m \leq b_d(X) + \frac{1}{2}(\chi(\mathbb{R}X^+) + \chi(\mathbb{R}X^-)) - 2 + \delta(\mathbb{C}X^+) + \delta(\mathbb{C}X^-)$$

where $N_d^m = N_d^m = \sum_{k=0}^d \binom{m-1}{k} = \binom{m-2}{d} + 2 \sum_{k=0}^{d-1} \binom{m-2}{k}$ is the number of the partition regions for a generic arrangement of m hyperplanes in $\mathbb{R}P^d$. It is easy to check that $b_d(X) = \binom{m-2}{d} + 1$ for even d (while $b_d(X) = \binom{m-2}{d}$ for odd d). Now the obvious relation $\chi(\mathbb{R}X^+) + \chi(\mathbb{R}X^-) = 2\chi(\mathbb{R}P^d) = 2$ and the estimate of $\delta(\mathbb{C}X^{\pm})$ in the following proposition show that (6-2) is indeed an equality and thus all the intermediate estimates, including (6-1), are also equalities.

6.1.3. Proposition. $\delta(\mathbb{C}X^{\pm}) \leq N_{d-1}^{m-1} = \sum_{0 \leq k \leq d-1} \binom{m-2}{k}$.

The proof is given in subsection 7.4. \square

6.2. Arnold inequalities for quasi-cuspidal surfaces. A real surface $\mathbb{C}X$ will be called *quasi-cuspidal* if the links, $\mathbb{R}M_x$, for $x \in \mathbb{R}X$ are circles, i.e., $\mathbb{R}X$ is topologically non-singular. We call a connected component $\Gamma \subset \mathbb{R}X$ *elliptic, parabolic or hyperbolic* if $-\langle \Gamma, \Gamma \rangle_{\mathbb{C}X}$ is positive, zero or negative respectively. Such terms are motivated by the relation $-\langle \Gamma, \Gamma \rangle_{\mathbb{C}X} = \chi(\Gamma)$ satisfied provided $\Gamma \cap \text{Sing}(\mathbb{R}X) = \emptyset$. More generally, if $\Gamma \cap \text{Sing}(\mathbb{R}X)$ contains only QI-singularities, Theorem 2.2.4 shows that one can treat $-\langle \Gamma, \Gamma \rangle_{\mathbb{C}X}$ as the *weighted Euler characteristic* of Γ , in which points $x \in \Gamma \cap \text{Sing}(\mathbb{R}X)$ are counted with the weights $-\lambda_x([\mathbb{R}M_x], [\mathbb{R}M_x])$.

Given a real quasi-cuspidal $\overline{\mathbb{Q}I}$ -surface $\mathbb{C}X$, denote by c^+ , c^0 , c^- the number of *oriented* elliptic, parabolic and hyperbolic components of $\mathbb{R}X$. Inequalities of Theorem 4.2.1 give estimates for c^+ , c^0 , c^- . (4-17) yields $c^- \leq p_g(\mathbb{C}X^{\text{res}})$, so, for instance, $c^- = 0$ for a singular rational or Enriques surface. If $\mathbb{C}X$ is a quasi-cuspidal surface with only $\mathbb{Z}/2$ -inessential singularities, then Theorem 4.2.1 implies that

$$c^0 + c^- \leq \chi_a(\mathbb{C}X^{\text{res}}) + b_1(\mathbb{C}X^{\text{res}}; \mathbb{Z}/2)$$

which gives for instance $c^0 \leq 1$ for rational surfaces and $c^0 \leq 2$ for Enriques surfaces. The estimates (4-19) and (4-20) can be used similarly and give an information about $\chi(\mathbb{R}X)$, like in the non-singular case.

Note that none of the above estimate follows automatic from the well known similar estimates for non-singular surfaces, because c^- , c^0 (as well as c^+) may decrease (as well as increase) after we pass to a resolution $\mathbb{C}X^{\text{res}} \rightarrow \mathbb{C}X$. As a simplest example, take a double plane, $\mathbb{C}X^+$, branched along a quartic $\mathbb{C}A \subset \mathbb{C}P^2$, with $\mathbb{R}A$ being a single oval with 3 ordinary cusps. $\mathbb{R}X^+$ has a parabolic component, which becomes elliptic after resolution.

Remark. The generalized Arnold-Viro inequalities for quasi-cuspidal surfaces have extremal properties analogous to the well-known such properties for non-singular

surfaces (see [Wi] [V4]). Namely, the estimates (4-18), (4-20) can be improved by 1, unless all the components of $\mathbb{R}X$ are orientable and parabolic. More generally, if (4-18) or (4-20) is an equality for any $\overline{\mathbb{Q}\mathbb{I}}$ -surface $\mathbb{C}X$, then there exists an integral lifting, $a \in H_2(\mathbb{R}X)$, of the fundamental class $[\mathbb{R}X]_2 \in H_2(\mathbb{R}X; \mathbb{Z}/2)$, of $\mathbb{R}X$, such that the image of a in $H_2(\mathbb{C}X')$ (and thus, in $H_2(\mathbb{C}X)$) vanishes. This gives certain restriction on the intersections (compare with the analogous formulation in [Kh2], where the case of nodal curves is considered).

6.3. Arnold inequalities for cuspidal and quasi-cuspidal curves. A topologically non-singular complex curve will be called a *cuspidal curve*; by a *quasi-cuspidal curve* we mean a real curve, $\mathbb{C}A$, with the topologically non-singular real part $\mathbb{R}A$. Accordingly, the singularities of a cuspidal curve will be called (generalized) *cusps*, and the real singularities of a quasi-cuspidal curve, respectively, *quasi-cusps*. The Smith inequality (7-2) implies that cusps are $\mathbb{Q}\mathbb{I}^S$ -singularities (and, moreover, $\mathbb{Z}/2$ -inessential singularities).

Like for non-singular curves, the connected components of $\mathbb{R}A$ for a quasi-cuspidal real curve $\mathbb{C}A \subset \mathbb{C}\mathbb{P}^2$ are all null-homotopic in $\mathbb{R}\mathbb{P}^2$ if $\mathbb{C}A$ has even degree $d = 2k$. Following the tradition, we call null-homotopic components *ovals*. An oval is called even (or odd), if it lies inside an even (respectively, odd) number of the other ovals.

Consider the double plane, $\pi: X \rightarrow \mathbb{C}\mathbb{P}^2$, branched along $\mathbb{C}A$ and let $\mathbb{C}X^\pm$ denote X endowed with the complex conjugation conj^\pm , covering the complex conjugation in $\mathbb{C}\mathbb{P}^2$, and put $\mathbb{R}\mathbb{P}_\pm^2 = \pi(\mathbb{R}X^\pm)$. Like in the non-singular case, there exists a single non-orientable partition component of $\mathbb{R}\mathbb{P}^2 - \mathbb{R}A$. We denote it by W_∞ , and let $\Gamma_\infty = \pi^{-1}(W_\infty)$, assuming for definiteness that $\text{sign}(W_\infty) = -$. Given a $\overline{\mathbb{Q}\mathbb{I}}$ -curve $\mathbb{C}A$, we denote by p^+ , p^- and p^0 , (n^+ , n^- and n^0) the number of even (respectively, odd) ovals, $C_i \subset \mathbb{R}A$, with the elliptic, parabolic and hyperbolic component $\Gamma_i \subset \mathbb{R}X^+$ (respectively, $\Gamma_i \subset \mathbb{R}X^-$), where $\Gamma_i = \pi^{-1}(W_i)$ and W_i is the partition region bounded by C_i from outside. We introduce furthermore the indicators, $\varepsilon^+, \varepsilon^0, \varepsilon^- \in \{0, 1\}$, showing the type of the components Γ_∞ . If k is even and thus Γ_∞ is non-orientable, we put $\varepsilon^+ + \varepsilon^0 + \varepsilon^- = 0$. If k is odd, then $\varepsilon^+ + \varepsilon^0 + \varepsilon^- = 1$ with the non-vanishing indicator being ε^+ , ε^0 or ε^- , if Γ_∞ is elliptic, parabolic, or hyperbolic respectively. The terms r, ν, μ^\pm , are defined as in §4.

6.3.1. Theorem. *Assume that $\mathbb{C}A \subset \mathbb{C}\mathbb{P}^2$ is a real quasi-cuspidal $\mathbb{Q}\mathbb{I}$ -curve of degree $2k$. Then*

$$\begin{aligned} n^- + n^0 + \varepsilon^- &\leq \frac{1}{2}(k-1)(k-2) - \frac{1}{2}\mu^+ + \min(n^0, r - \nu - 1) \\ p^- + p^0 &\leq \frac{1}{2}(k-1)(k-2) - \frac{1}{2}\mu^+ + \min(p^0, r - \nu - \frac{1}{2}(1 - (-1)^k)) \\ n - p^- &\leq \frac{3}{2}k(k-1) - \frac{1}{2}\mu^- + \min(p^0, r - \nu - \frac{1}{2}(1 - (-1)^k)) \\ p - n^- + \varepsilon_+ &\leq \frac{3}{2}k(k-1) - \frac{1}{2}\mu^- + \min(n^0 + 1, r - \nu) \end{aligned}$$

In the case of cuspidal curves, we have $r = 1, \nu = 0$ and the above inequalities are simplified as follows

6.3.2. Corollary. *If $\mathbb{C}A \subset \mathbb{C}P^2$ is a real cuspidal curve, then*

$$(6-3) \quad \begin{aligned} n^- + n^0 + \varepsilon^- &\leq \frac{1}{2}(k-1)(k-2) - \frac{1}{2}\mu^+ \\ p^- + p^0 &\leq \frac{1}{2}(k-1)(k-2) - \frac{1}{2}\mu^+ + \frac{1}{2}(1 + (-1)^k) \end{aligned}$$

$$(6-4) \quad \begin{aligned} n - p^- &\leq \frac{3}{2}k(k-1) - \frac{1}{2}\mu^- + \frac{1}{2}(1 + (-1)^k) \\ p - n^- + \varepsilon^+ &\leq \frac{3}{2}k(k-1) - \frac{1}{2}\mu^- + 1 \end{aligned}$$

Remarks. (1) The extremal properties of the Arnold inequalities mentioned in 6.2 allow us, like in the case of non-singular $\mathbb{C}A$, improve by 1 some of the above estimates, unless the curve $\mathbb{R}A$ has a rather special topology, with all the components $\Gamma \subset \mathbb{R}X^\pm$ being parabolic; for instance, one can put 1 instead of $\frac{1}{2}(1 + (-1)^k)$ in Theorem 6.3.1 and to drop $\frac{1}{2}(1 + (-1)^k)$ from (6-3) and (6-4).

(2) The term $\frac{1}{2}(1 + (-1)^k)$ can be omitted in (6-4) if $\mathbb{C}A$ is non-singular (cf. [Wi, V4]), since in the extremal case we get $p = p^0 = n$ and (6-3) turns out to be stronger than (6-4). Similar arguments show that we can also omit this term if $\mu^- - \mu^+ \leq 2k^2 - 2$.

6.4. Pentics with three A_4 -cusps. The following example can give an idea how one can get an information about the geometry of the singularities on a curve. Assume that $\mathbb{C}A$ is a real pentic having three cusps, $x_i \in \mathbb{R}A$, $i = 1, 2, 3$, of the type A_4 (so that $\mathbb{C}A$ is rational). Consider a line $\mathbb{C}L$ passing through the points x_1, x_2 , and let y denote the third intersection point of $\mathbb{R}A$ with $\mathbb{R}L$.

6.4.1. Proposition. *The curve $\mathbb{R}A$ approaches the line $\mathbb{R}L - \{y\}$ at points x_1, x_2 from the opposite sides (which makes sense, because $\mathbb{R}L - \{y\}$ is bilateral in $\mathbb{R}P^2 - \{y\}$).*

Proof. The partition form of a sextic $\mathbb{C}A' = \mathbb{C}A \cup \mathbb{C}L$ is singular by Corollary 4.3.3, since it is reducible. However, an elementary combinatorial analysis of the possible mutual position of the cusps and $\mathbb{R}L$ shows that this form cannot be singular if $\mathbb{R}A$ comes to $\mathbb{R}L - \{y\}$ from the same side at x_1 and x_2 .

Remark. Proposition 6.4.1 can be easily derived also from Gudkov's theorem characterizing possible mutual position of a non-singular pentic and a line, however, our arguments are essentially more elementary than the Gudkov's result.

§7 APPENDIX A. THE SMITH THEORY ESTIMATES

7.1. The Smith sequence. Given an involution on a finite CW -complex, $c: Z \rightarrow Z$, with the fixed point set F and the quotient Z/c , we can write a long homology *Smith exact sequence* (with $\mathbb{Z}/2$ -coefficients)

$$\cdots \rightarrow H_{k+1}(Z/c, F) \rightarrow H_k(Z/c, F) \oplus H_k(F) \rightarrow H_k(Z) \rightarrow H_k(Z/c, F) \rightarrow \cdots$$

where $H_k(F) \rightarrow H_k(Z)$ is the inclusion homomorphism, $H_k(Z) \rightarrow H_k(Z/c, F)$ is induced by the quotient map $q: Z \rightarrow Z/c$, and $H_{k+1}(Z/c, F) \rightarrow H_k(F)$ is taken from the homology sequence of the pair $(Z/c, F)$ (see, e.g., [Br, p.123], or [Wi, Appendix]). Denote by ν_k the rank of the inclusion homomorphism $H_k(F; \mathbb{Z}/2) \rightarrow H_k(Z/c; \mathbb{Z}/2)$.

Analyzing this sequence one easily obtains the following estimates, called *the Smith inequalities*.

7.1.1. Theorem. For any $k \geq 0$

$$(7-1) \quad b_k(Z/c, F; \mathbb{Z}/2) + b_k(F; \mathbb{Z}/2) \leq b_{k+1}(Z/c, F; \mathbb{Z}/2) + b_k(Z; \mathbb{Z}/2)$$

For any $l \geq 0$

$$(7-2) \quad b_l(Z/c, F; \mathbb{Z}/2) + \sum_{k \geq l} b_k(F; \mathbb{Z}/2) \leq \sum_{k \geq l} b_k(Z; \mathbb{Z}/2)$$

and in particular $b_*(F; \mathbb{Z}/2) \leq b_*(Z; \mathbb{Z}/2)$, where b_* stands for the sum of all Betti numbers.

$$(7-3) \quad b_k(Z; \mathbb{Z}/2) \leq b_k(Z/c; \mathbb{Z}/2) + b_k(Z/c, F; \mathbb{Z}/2)$$

for any $k \geq 0$. The latter can be also formulated as

$$(7-4) \quad b_k(Z; \mathbb{Z}/2) \leq 2b_k(Z/c; \mathbb{Z}/2) + b_{k-1}(F; \mathbb{Z}/2) - \nu_k - \nu_{k-1}$$

where by definition $\nu_{-1} = b_{-1}(\dots) = 0$.

Here (7-1) is proved obviously, adding (7-1) for all $k \geq l$, we obtain (7-2). (7-3) follows from that the dimension of the kernel of $H_k(Z/c, F; \mathbb{Z}/2) \rightarrow H_{k-1}(F; \mathbb{Z}/2)$ and of the cokernel of $H_{k+1}(Z/c, F; \mathbb{Z}/2) \rightarrow H_k(F; \mathbb{Z}/2)$ in the Smith sequence give in sum $b_k(Z/c; \mathbb{Z}/2)$. (7-4) is obtained from (7-3) if we use the expression of $b_k(Z/c, F; \mathbb{Z}/2)$ given by the exact sequence of the pair $(Z/c, F)$.

Remarks.

- (1) Using augmented homology groups, we obtain the same inequalities for \tilde{b}_k .
- (2) Assume that $\dim Z = m$, $b_m(Z; \mathbb{Z}/2) = 1$ and $\dim F \leq m - 2$. Then, analyzing the first non-trivial terms in the Smith sequence, we obtain

$$(7-5) \quad \begin{aligned} b_m(Z/c; \mathbb{Z}/2) &= 1 \\ b_{m-1}(Z/c, F; \mathbb{Z}/2) &\leq b_{m-1}(Z; \mathbb{Z}/2) + 1 \\ b_{m-1}(Z; \mathbb{Z}/2) &\leq b_{m-1}(Z/c; \mathbb{Z}/2) + b_{m-1}(Z/c, F; \mathbb{Z}/2) - 1 \\ &= 2b_{m-1}(Z/c; \mathbb{Z}/2) + \tilde{b}_{m-2}(F; \mathbb{Z}/2) - 1 \end{aligned}$$

where (7-5) is an improvement of (7-3)–(7-4) in this special case

- (3) If $A \subset Z$ is a c -invariant subcomplex, then the Smith sequence for the induced involution $Z/A \rightarrow Z/A$ gives

$$\dots \rightarrow H_k(Z/c, F \cup A/c) \oplus H_k(F, F \cap A) \rightarrow H_k(Z, A) \rightarrow H_k(Z/c, F \cup A/c) \rightarrow \dots$$

(all the groups are with $\mathbb{Z}/2$ -coefficients), which gives obvious relative versions of the estimates (7-1)–(7-4).

7.2. Proof of Lemma 3.1.3. The homology sequences of $(\overline{X}, \mathbb{R}X)$ and the universal coefficients formula imply

$$(7-6) \quad \delta(\mathbb{C}X) = b_{d+1}(\overline{X}, \mathbb{R}X) - b_{d+1}(\overline{X}) \leq b_{d+1}(\overline{X}, \mathbb{R}X; \mathbb{Z}/2) - b_{d+1}(\overline{X})$$

Applying (7-2), with $l = d + 1$, to the complex conjugation in $Z = \mathbb{C}X$ and using (7-6) we obtain estimate (3-5). The next estimate, (3-6), can be obtained from (3-5), since for a d -dimensional complete intersection, $\mathbb{C}X$, with isolated singularities, we have $b_k(\mathbb{C}X) = b_k(\mathbb{C}P^d)$, for all k , except possibly d and $d + 1$. The latter follows from the Lefschetz hyperplane section theorem, homotopy equivalence of the Milnor fibers to wedges of spheres, and from the exact sequence of the pair $(\mathbb{C}X^\tau, U^\tau)$.

We may prove (3-7) in a slightly more general setting, with $\mathbb{C}P^d$ being replaced by a real non-singular variety $\mathbb{C}P$ of dimension d . Applying (7-3) to the deck transformation of the double covering $p: \mathbb{C}X \rightarrow \mathbb{C}P$ branched along a real reduced hypersurface $\mathbb{C}A \subset \mathbb{C}P$, we obtain

$$(7-7) \quad b_k(\mathbb{C}X; \mathbb{Z}/2) \leq b_k(\mathbb{C}P; \mathbb{Z}/2) + b_k(\mathbb{C}P, \mathbb{C}A; \mathbb{Z}/2)$$

which can be improved by 1 for $k = 2d$ and $k = 2d - 1$ as (7-5) shows. Using the improved inequality to estimate $b_k(\mathbb{C}X; \mathbb{Z}/2)$ in (3-5), we obtain (3-7). Similarly, (7-4) combined with (3-6) gives (3-8).

7.3. Proof of Lemma 4.1.4. The relations $b_i(\overline{X}) = \frac{1}{2}b_i(\mathbb{C}X)$, $i = 1, 3$, in (4-10) and (4-11) follow, for instance, from that the mixed Hodge structure in $H^i(\mathbb{C}X; \mathbb{C})$ is pure. There is a more elementary way to derive this relations: just to notice that $b_1(\mathbb{C}X) = b_1(\mathbb{C}X^\tau)$, $b_1(\overline{X}) = b_1(\overline{X}^\tau)$, on one hand, whereas $b_3(\mathbb{C}X) = b_3(\mathbb{C}X^{\text{res}})$, $b_3(\overline{X}) = b_3(\overline{X}^{\text{res}})$, on the other hand, and then to use the relation $b_i(\overline{Z}) = \frac{1}{2}b_i(\mathbb{C}Z)$, which obviously holds for non-singular real varieties, $\mathbb{C}Z$, and odd i (cf. 8.1). The relation (7-4) applied to the deck transformation of $\mathbb{C}X \rightarrow \mathbb{C}P$ and its restriction to $\mathbb{C}X'$ and $\mathbb{C}X''$ gives

$$(7-8) \quad b_1(\mathbb{C}X; \mathbb{Z}/2) \leq 2b_1(\mathbb{C}P; \mathbb{Z}/2) + \tilde{b}_0(\mathbb{C}A)$$

$$(7-9) \quad b_3(\mathbb{C}X; \mathbb{Z}/2) \leq 2b_3(\mathbb{C}P; \mathbb{Z}/2) + b_2(\mathbb{C}A) - \nu$$

$$(7-10) \quad \begin{aligned} b_1(\mathbb{C}X'; \mathbb{Z}/2) &\leq 2b_1(\mathbb{C}P; \mathbb{Z}/2) + \tilde{b}_0(\mathbb{C}A') \\ b_1(\mathbb{C}X''; \mathbb{Z}/2) &\leq 2b_1(\mathbb{C}P; \mathbb{Z}/2) + \tilde{b}_0(\mathbb{C}A'') \end{aligned}$$

where $\nu = \text{rk}(H_2(\mathbb{C}A; \mathbb{Z}/2) \rightarrow H_2(\mathbb{C}P; \mathbb{Z}/2))$, like in 4.1. (7-8) implies (4-10), since $b_1(\mathbb{C}X) \leq b_1(\mathbb{C}X; \mathbb{Z}/2)$. (4-11) contains 3 + 1 estimates of $b_3(\overline{X})$, and the first among them follows from (7-9). The next two estimates follows from

$$b_3(\overline{X}) = b_3(\overline{X}', \partial \overline{X}') = b_1(\overline{X}') \leq b_1(\mathbb{C}X') \leq 2b_1(\mathbb{C}P; \mathbb{Z}/2) + \tilde{b}_0(\mathbb{C}A')$$

$$2b_3(\overline{X}) = b_3(\mathbb{C}X) = b_3(\mathbb{C}X'', \partial \mathbb{C}X'') = b_1(\mathbb{C}X'') \leq 2b_1(\mathbb{C}P; \mathbb{Z}/2) + \tilde{b}_0(\mathbb{C}A'')$$

The fourth estimate is a combination of (7-8) with the inequality $b_3(\overline{X}) \leq b_1(\overline{X}) + \beta$, which follows from (4-21). The arguments in 7.2 give in the case $d = 2$

$$\delta(\mathbb{C}X) \leq 2b_3(\mathbb{C}P; \mathbb{Z}/2) + b_2(\mathbb{C}A) - \nu - b_3(\overline{X})$$

$$\delta(\mathbb{C}X') \leq 2b_3(\mathbb{C}P'; \mathbb{Z}/2) + b_2(\mathbb{C}A') - \nu' - b_3(\overline{X}')$$

Furthermore, for $\alpha_{\text{Im}} \geq 1$ we have

$$b_3(\overline{X}') = b_1(\overline{X}', \partial \overline{X}') = b_1(\overline{X}) + \tilde{b}_0(\partial \overline{X}) = b_1(\overline{X}) + (\alpha_{\text{Im}} - 1)$$

$$b_3(\mathbb{C}P'; \mathbb{Z}/2) = b_1(\mathbb{C}P', \partial \mathbb{C}P'; \mathbb{Z}/2) = b_1(\mathbb{C}P; \mathbb{Z}/2) + \tilde{b}_0(\partial \mathbb{C}P') = b_1(\mathbb{C}P; \mathbb{Z}/2) + (2\alpha_{\text{Im}} - 1)$$

which proves (4-12). (4-13) follows from (4-12) combined with (4-7).

7.4. Proof of Lemma 4.2.3. Let $\mathbb{C}U_{\text{Im}}^{(2)}$ denote the union of the connected components of $\mathbb{C}U$ around the $\mathbb{Z}/2$ -essential imaginary singular points of $\mathbb{C}X$. Put $\mathbb{C}X_{\text{Im}}^{(2)} = \text{Cl}(\mathbb{C}X - \mathbb{C}U_{\text{Im}}^{(2)})$. We have $H_2(\overline{X}'; \mathbb{Q}) \cong H_2(\overline{X}_{\text{Im}}^{(2)}; \mathbb{Q})$, since $\overline{X}_{\text{Im}}^{(2)}$ is obtained from \overline{X}' by removing several cones over \mathbb{Q} -homology spheres, and thus $\delta(\mathbb{C}X') = \dim \ker(H_2(\mathbb{R}X; \mathbb{Q}) \rightarrow H_2(\overline{X}_{\text{Im}}^{(2)}; \mathbb{Q})) = b_3(\overline{X}_{\text{Im}}^{(2)}, \mathbb{R}X) - b_3(\overline{X}_{\text{Im}}^{(2)})$. Assume that $\alpha_{\text{Im}}^{(2)} \geq 1$, then the universal coefficients formula combined with (7-2) implies that $b_3(\overline{X}_{\text{Im}}^{(2)}, \mathbb{R}X) \leq b_3(\overline{X}_{\text{Im}}^{(2)}, \mathbb{R}X; \mathbb{Z}/2) \leq b_3(\mathbb{C}X_{\text{Im}}^{(2)}; \mathbb{Z}/2)$ and the duality gives $b_3(\overline{X}_{\text{Im}}^{(2)}) = b_1(\overline{X}_{\text{Im}}^{(2)}; \partial \overline{X}_{\text{Im}}^{(2)}) = b_1(\overline{X}) + \alpha_{\text{Im}}^{(2)} - 1$.

Consider an intermediate resolution, $\mathbb{C}X_{\text{Im}}^{\text{res}} \rightarrow \mathbb{C}X$, which resolves the imaginary $\mathbb{Z}/2$ -essential singularities, and let $\mathbb{C}X^{\text{res}} \rightarrow \mathbb{C}X_{\text{Im}}^{\text{res}}$ resolves all the other singularities. Viewing $\mathbb{C}X_{\text{Im}}^{(2)}$ as a subset of $\mathbb{C}X_{\text{Im}}^{\text{res}}$, we obtain from the homology sequence of $(\mathbb{C}X_{\text{Im}}^{\text{res}}, \mathbb{C}X_{\text{Im}}^{(2)})$ that

$$b_3(\mathbb{C}X_{\text{Im}}^{(2)}; \mathbb{Z}/2) \leq b_3(\mathbb{C}X_{\text{Im}}^{\text{res}}; \mathbb{Z}/2) + 2\alpha_{\text{Im}}^{(2)} - 1$$

and thus

$$\delta(\mathbb{C}X') \leq b_3(\mathbb{C}X_{\text{Im}}^{\text{res}}; \mathbb{Z}/2) - b_1(\overline{X}) + \alpha_{\text{Im}}^{(2)}$$

Denote by $\mathbb{C}E^{(2)}$ the exceptional divisor of the resolution $\mathbb{C}X^{\text{res}} \rightarrow \mathbb{C}X_{\text{Im}}^{\text{res}}$ and let $\theta = \dim \ker(H_2(\mathbb{C}E^{(2)}; \mathbb{Z}/2) \rightarrow H_2(\mathbb{C}X^{\text{res}}; \mathbb{Z}/2))$. The homology sequence of $(\mathbb{C}X^{\text{res}}, \mathbb{C}E^{(2)})$ gives

$$b_3(\mathbb{C}X_{\text{Im}}^{\text{res}}; \mathbb{Z}/2) = b_3(\mathbb{C}X^{\text{res}}, \mathbb{C}E^{(2)}; \mathbb{Z}/2) = b_3(\mathbb{C}X^{\text{res}}; \mathbb{Z}/2) + \theta$$

Analyzing the homology sequence of $(\mathbb{C}U_{\mathfrak{R}}, \partial \mathbb{C}U_{\mathfrak{R}})$, where $\mathbb{C}U_{\mathfrak{R}}$ is the union of the components of $\mathbb{C}U$ around the real singularities of $\mathbb{C}X$, we obtain an estimate $\theta \leq d_{\mathfrak{R}}^{(2)}$, (here we use that $\mathbb{Z}/2$ -essential singularities of $\mathbb{C}X_{\text{Im}}^{\text{res}}$ are real). Thus,

$$(7-11) \quad \delta(\mathbb{C}X') \leq b_3(\mathbb{C}X^{\text{res}}; \mathbb{Z}/2) - b_1(\overline{X}) + \alpha_{\text{Im}}^{(2)} + d_{\mathfrak{R}}^{(2)}$$

In the case $\alpha_{\text{Im}}^{(2)} = 0$, the same arguments bring an analogous formula, in which 1 stands instead of $\alpha_{\text{Im}}^{(2)}$. This gives (4-24), because $b_1(\overline{X}) = \frac{1}{2}b_1(\mathbb{C}X)$ (cf. 7.3). (4-25) follows from (7-11), (4-21) and the remark in the beginning of 7.3.

7.5. Proof of Proposition 6.1.3. Along with the arrangement, $\mathbb{C}A = \mathbb{C}A_1 \cup \dots \cup \mathbb{C}A_m$, of hyperplanes $\mathbb{C}A_i \subset \mathbb{C}P^d$ in generic position and the double covering $X \rightarrow \mathbb{C}P^d$ branched along $\mathbb{C}A$, considered in 6.1, we consider the affine arrangement $\mathbb{C}A_a = \mathbb{C}A \setminus \mathbb{C}A_m$ in $\mathbb{C}^d = \mathbb{C}P^d - \mathbb{C}A_m$ and the double covering $X_a \rightarrow \mathbb{C}^d$ branched along $\mathbb{C}A_a$. Endowed with one of the two real structures covering the real structure in \mathbb{C}^d , X_a becomes a real affine variety, $\mathbb{C}X_a^\varepsilon \subset \mathbb{C}X^\varepsilon$, where $\varepsilon = +$ or $\varepsilon = 1$. Put

$$\begin{aligned} \delta(\mathbb{C}X_a^\varepsilon) &= \dim \ker(H_d(\mathbb{R}X_a^\varepsilon; \mathbb{Q}) \rightarrow H_d(X_a; \mathbb{Q})) \\ \delta_2(\mathbb{C}X_a^\varepsilon) &= \dim \ker(H_d(\mathbb{R}X_a^\varepsilon; \mathbb{Z}/2) \rightarrow H_d(X_a; \mathbb{Z}/2)) \end{aligned}$$

7.5.1. Lemma. $\delta(\mathbb{C}X_a) = \delta_2(\mathbb{C}X_a) = 0$

Proof. We obviously have

$$\begin{aligned}\delta(\mathbb{C}X_a^\varepsilon) &= \dim \ker(H_d(\mathbb{R}X_a^\varepsilon; \mathbb{Q}) \rightarrow H_d(\overline{X}_a^\varepsilon; \mathbb{Q})) = b_{d+1}(\overline{X}_a^\varepsilon, \mathbb{R}X_a^\varepsilon) - b_{d+1}(\overline{X}_a^\varepsilon) \\ \delta_2(\mathbb{C}X_a^\varepsilon) &\leq \dim \ker(H_d(\mathbb{R}X_a^\varepsilon; \mathbb{Z}/2) \rightarrow H_d(\overline{X}_a^\varepsilon; \mathbb{Z}/2)) = b_{d+1}(\overline{X}_a^\varepsilon, \mathbb{R}X_a^\varepsilon; \mathbb{Z}/2) - b_{d+1}(\overline{X}_a^\varepsilon; \mathbb{Z}/2)\end{aligned}$$

where $b_{d+1}(\overline{X}_a^\varepsilon, \mathbb{R}X_a^\varepsilon) \leq b_{d+1}(\overline{X}_a^\varepsilon, \mathbb{R}X_a^\varepsilon; \mathbb{Z}/2)$ by the universal coefficients formula. So, it is enough to prove that $b_{d+1}(\overline{X}_a^\varepsilon, \mathbb{R}X_a^\varepsilon; \mathbb{Z}/2)$ vanishes. Vanishing follows from the inequalities

$$(7-13) \quad b_{d+1}(\overline{X}_a^\varepsilon, \mathbb{R}X_a^\varepsilon; \mathbb{Z}/2) \leq \sum_{k \geq 1} b_{d+k}(X_a; \mathbb{Z}/2) \leq 2 \sum_{k \geq 1} b_{d+k}(\mathbb{C}^d; \mathbb{Z}/2) + \sum_{k \geq 0} b_{d+k}(\mathbb{C}A_a; \mathbb{Z}/2) = 0$$

The first of these inequalities follows from (7-2) applied to the complex conjugation, the second one follows from (7-4) applied to the covering transform of $X_a \rightarrow \mathbb{C}^d$, and $b_{d+k}(\mathbb{C}A_a; \mathbb{Z}/2) = 0$ for $k \geq 0$ because $\mathbb{R}A_a$ is a deformational retract of $\mathbb{C}A_a$. \square

To derive Lemma 6.1.2 we note that the inclusion homomorphism $H_d(X_a) \rightarrow H_d(X)$ is monomorphic, because $H_{d+1}(X, X_a) \cong H^{d-1}(\mathbb{C}A_m)$ by the Alexander duality in \mathbb{Q} -homology manifold X and $H^{d-1}(\mathbb{C}A_m) = H^{d-1}(\mathbb{C}P^d) = 0$ for even d . Vanishing of $\delta(\mathbb{C}X_a^\varepsilon)$ implies linear independence of the fundamental classes, $[\Gamma_i] \in H_d(\mathbb{C}X^\varepsilon)$, of $\Gamma_i = \pi^{-1}(W_i)$, for those of the partition components, $W_i \subset \mathbb{R}P_\varepsilon^d$, which do not have points in common with $\mathbb{R}A_m$. So, $\delta(\mathbb{C}X^\varepsilon)$ cannot exceed the number of the other components, $W_i \subset \mathbb{R}P_\varepsilon^d$, such that $W_i \cap \mathbb{R}A_m \neq \emptyset$. This number is obviously N_{d-1}^{m-1} , for a generic hyperplane arrangement.

Remark. The most of the above arguments can be applied (after a simple modification) to non-generic arrangement as well. The exception is the argument in the last paragraph, which uses the duality in X and thus requires that $\mathbb{C}A$ is a $\overline{\mathbb{Q}\mathbb{I}}^S$ -hypersurface.

§8. APPENDIX B: APPLICATION OF THE HODGE THEORY TO REAL ALGEBRAIC VARIETIES

8.1. The Hodge structure in real QI-varieties. For QI-varieties the mixed Hodge structure in $H^*(\mathbb{C}X; \mathbb{C})$ is known to be pure, which gives the usual Hodge splitting $H^*(\mathbb{C}X; \mathbb{C}) = \bigoplus_{p,q \geq 0} H^{p,q}(\mathbb{C}X)$ (indeed, the non-pure part of $H^*(\mathbb{C}X; \mathbb{C})$ is the kernel of $\text{res}^*: H^*(\mathbb{C}X; \mathbb{Q}) \rightarrow H^*(\mathbb{C}X^{\text{res}}; \mathbb{Q})$ induced by a resolution $\text{res}: \mathbb{C}X^{\text{res}} \rightarrow \mathbb{C}X$; the latter is a degree 1 map of rational homology manifolds, and thus res^* is a monomorphism). An embedding $\mathbb{C}X \subset \mathbb{C}P^N$ gives a hyperplane class, $h \in H^{1,1}(\mathbb{C}X)$, and defines a Lefschetz decomposition like in non-singular case. To define the subspaces $P^{p,q}(\mathbb{C}X) \subset H^{p,q}(\mathbb{C}X)$ of primitive (p,q) -classes in $\mathbb{C}X$, one can use the inner product in $H^*(\mathbb{C}X; \mathbb{C})$ induced from $H^*(\mathbb{C}X^{\text{res}}; \mathbb{C})$ via res^* . The Hodge index theorem for $\mathbb{C}X^{\text{res}}$ obviously descends to $\mathbb{C}X$, since res^* preserves the Hodge filtration.

This allow us to can reproduce the arguments applied in [Kh3] to the case of non-singular real varieties as follows. Since conj^* interchanges $H^{p,q}(\mathbb{C}X)$ and $H^{q,p}(\mathbb{C}X)$,

both the trace and the signature of the involution conj , vanishes being restricted to $H^{p,q}(\mathbb{C}X) \oplus H^{q,p}(\mathbb{C}X)$, for $p \neq q$. Therefore,

$$\begin{aligned} \text{tr}|_{H^*(\mathbb{C}X;\mathbb{R})} &= \sum_{p=0}^{2n} \text{tr}|_{H^{p,p}(\mathbb{C}X)} \\ \sigma_{\text{conj}}|_{H^{2n}(\mathbb{C}X;\mathbb{R})} &= \sigma_{\text{conj}}|_{H^{n,n}(\mathbb{C}X)} \end{aligned}$$

Using that multiplication by h , $H^{p,p}(\mathbb{C}X) \rightarrow \cup h \gg H^{p+1,p+1}(\mathbb{C}X)$, maps ± 1 -eigenspace of conj^* into the ∓ 1 -eigenspace, we obtain

$$(8-1) \quad \text{tr}|_{H^*(\mathbb{C}X;\mathbb{R})} = \sum_{p=0}^n \text{tr}|_{P^{p,p}(\mathbb{C}X)}$$

$$(8-2) \quad \text{tr}|_{H^*(\mathbb{C}X;\mathbb{R})} = \text{tr}|_{H^d(\mathbb{C}X;\mathbb{R})} + 2 \sum_{p \geq 0} \text{tr}|_{P^{n-1-2p,n-1-2p}(\mathbb{C}X)}$$

$$(8-3) \quad \sigma_{\text{conj}}|_{H^{2n}(\mathbb{C}X;\mathbb{R})} = \sum_{p=0}^n \sigma_{\text{conj}}|_{P^{p,p}(\mathbb{C}X) \wedge h^{n-p}} = (-1)^n \sum_{p=0}^n \text{tr}|_{P^{p,p}(\mathbb{C}X)}$$

Here $\sigma_{\text{conj}}|_A$ denotes the signature of the involution, $\sigma(A^+) - \sigma(A^-)$, for a conj^* -invariant subspace $A \subset H^d(\mathbb{C}X;\mathbb{C})$ and the ± 1 -eigenspaces, A^\pm , of conj^* in A . To obtain (8-3) we used that the intersection form in $H^{2n}(\mathbb{C}X;\mathbb{R})$ is positive on $P^{p,q}(\mathbb{C}X) \wedge h^{n-\frac{1}{2}(p+q)}$ for even $\frac{1}{2}(p+q) \leq n$ and negative for odd, which follows from the analogous fact (the Hodge index theorem) for $\mathbb{C}X^{\text{res}}$.

It is convenient to formulate the above identities in terms of the functions $T^\pm(Z) = \chi(Z) \pm \sigma(Z)$, defined on compact rational homology manifolds, Z , of dimension $4n$, (possibly, with $\partial Z \neq \emptyset$), and functions $D^\pm(\mathbb{C}X) = T^\pm(\overline{X}) - \frac{1}{2}T^\pm(\mathbb{C}X)$, defined on real $\mathbb{Q}\text{I}$ -varieties of dimension $d = 2n$, and on conj -invariant compact codimension 0 (\mathbb{Q} -homology-)submanifolds of such varieties.

8.1.1. Theorem. *Assume that $\mathbb{C}X$ is a real algebraic $\mathbb{Q}\text{I}$ -variety of dimension $d = 2n$, $\varkappa = (-1)^n$. Then*

$$(8-4) \quad D^{-\varkappa}(\mathbb{C}X) = 0$$

$$(8-5) \quad D^{\varkappa}(\mathbb{C}X) = \chi(\mathbb{R}X)$$

$$(8-6) \quad b_d^{-\varkappa}(\overline{X}) = \frac{1}{2}(b_d^{-\varkappa}(\mathbb{C}X) - t(\mathbb{C}X))$$

$$(8-7) \quad b_d^{\varkappa}(\overline{X}) = \frac{1}{2}(b_d^{\varkappa}(\mathbb{C}X) + \chi(\mathbb{R}X) - t(\mathbb{C}X))$$

where $t(\mathbb{C}X) = \sum_{p \geq 0} \text{tr}|_{P^{n-1-2p,n-1-2p}(\mathbb{C}X)}$.

Proof of Theorem 8.1.1. (8-4) follows from (8-1) and (8-3). Together with the Lefschetz fixed point formula for the involution conj^* , which can be stated as $D^{-\varkappa}(\mathbb{C}X) + D^{\varkappa}(\mathbb{C}X) = \chi(\mathbb{R}X)$, it implies (8-5). (8-6) and (8-7) are versions of (8-4) and (8-5), which are obtained by comparing (8-2) with (8-3). \square

The Lefschetz hyperplane section theorem implies that $t(\mathbb{C}X) = t(\mathbb{CP}^d) = \kappa$, if $\mathbb{C}X$ is a complete intersection of dimension $d = 2n$. This gives the relations (3-3).

8.2. Proof of Theorem 3.2.2. One of the approaches to prove Theorem 3.2.2 is to use a version of the arguments in 8.1 applied to the mixed Hodge structure in the Milnor fiber, $\mathbb{C}U^\tau$ (using a similarity between the logarithm of the unipotent part of the monodromy in $H^d(\mathbb{C}U^\tau; \mathbb{C})$ and multiplication by h in $H^*(\mathbb{C}X; \mathbb{C})$). It is more elementary, however, to present another proof, which uses intersection (of the middle perversity) homology, rather than Mixed Hodge structure. Recall that the intersection homology groups of algebraic varieties have a pure Hodge structure, which is functorial, satisfy the usual properties (the Kähler package), including the both Lefschetz theorems, and Hodge index theorem [Sa], and coincides with the usual Hodge structure if $\mathbb{C}X$ is a \mathbb{Q} -homology manifold. This enables us to go through all the arguments of the previous subsection and obtain a version of Theorem 8.1.1 for the intersection homologies. For instance, a version of (8-4), that is needed for the proof of Theorem 3.2.2, can be formulated as follows. Denote by $\iota\chi$, $\iota\sigma$, the (middle perversity) intersection homology Euler characteristic and the signature of a pseudo-manifold and put $ID^\pm(\mathbb{C}X) = IT^\pm(\mathbb{C}X/\text{conj}) - \frac{1}{2}IT^\pm(\mathbb{C}X)$, where $IT^\pm(Z) = \iota\chi(Z) \pm \iota\sigma(Z)$. Here IT^\pm is well-defined for any stratified pseudo-manifold, Z , of dimension $4n$, whereas $\mathbb{C}X$ is supposed to be a real variety of dimension $\dim_{\mathbb{C}}(\mathbb{C}X) = 2n$, or conj-invariant compact codimension 0 (pseudo-)submanifold of such a variety.

8.2.1. Theorem. *Assume that $\mathbb{C}X$ is a real algebraic variety of dimension $d = 2n$, $\varkappa = (-1)^n$. Then $ID^{-\varkappa}(\mathbb{C}X) = 0$. \square*

Assume now that $\mathbb{C}X$ is a real complete intersection which has only isolated singularities. Let $\mathbb{C}X^\tau$ denote a real deformation of $\mathbb{C}X$ and $\mathbb{C}U$, \overline{U} , $\mathbb{C}U^\tau$ and \overline{U}^τ are chosen like in 3.3. By (8-4) and Theorem 8.2.1, $ID^{-\varkappa}(\mathbb{C}X) - D^{-\varkappa}(\mathbb{C}X^\tau) = 0$. On the other hand, additivity of $ID^{-\varkappa}$ and $D^{-\varkappa}$ together with $D^{-\varkappa}(\mathbb{C}X - \mathbb{C}U) = ID^{-\varkappa}(\mathbb{C}X - \mathbb{C}U)$ imply that $D^{-\varkappa}(\mathbb{C}U^\tau) - ID^{-\varkappa}(\mathbb{C}U) = 0$, or equivalently,

$$T^{-\varkappa}(\mathbb{C}U^\tau) - IT^{-\varkappa}(\mathbb{C}U) = 2(T^{-\varkappa}(\overline{U}^\tau) - IT^{-\varkappa}(\overline{U})).$$

Furthermore, we have

$$(8-8) \quad T^{-\varkappa}(\mathbb{C}U^\tau) = b_0(\mathbb{C}U^\tau) + 2b_d^{-\varkappa}(\mathbb{C}U^\tau) + b_d^0(\mathbb{C}U^\tau)$$

$$(8-9) \quad IT^{-\varkappa}(\mathbb{C}U) = b_0(\mathbb{C}U) - b_{d-1}(\partial \mathbb{C}U)$$

where (8-8) follows from that the connected component of $\mathbb{C}U^\tau$ (homeomorphic to the corresponding Milnor fibers of $\mathbb{C}X$) are homotopy equivalent to wedges of d -spheres. (8-9) follows from vanishing of the intersection “Betti numbers”, $Ib_k(\mathbb{C}U)$, for $k \geq d$, from that $Ib_k(\mathbb{C}U) = b_k(\partial \mathbb{C}U)$ for $k \leq d$, [GM1, p. 209], and from vanishing of $\tilde{b}_k(\partial \mathbb{C}U)$ for all $k < d - 1$ for ICIS.

The exact homology sequence of the pair $(\mathbb{C}U^\tau, \partial \mathbb{C}U^\tau)$ implies that $b_d^0(\mathbb{C}U^\tau) = b_{d-1}(\partial \mathbb{C}U^\tau)$, which gives $T^{-\varkappa}(\mathbb{C}U^\tau) - IT^{-\varkappa}(\mathbb{C}U) = 2(b_d^{-\varkappa}(\mathbb{C}U^\tau) + b_d^0(\mathbb{C}U^\tau))$, since $\partial \mathbb{C}U^\tau \cong \partial \mathbb{C}U$. The same arguments can be applied to \overline{U} , due to Lemma 3.2.1; thus $T^{-\varkappa}(\overline{U}^\tau) - IT^{-\varkappa}(\overline{U}) = 2b_d^{-\varkappa}(\overline{U}^\tau) + 2b_d^0(\overline{U}^\tau)$, which proves (3-12) in Theorem 3.2.2. (3-13) follows from (3-12) combined with the Lefschetz formula for the involution $\text{conj}|_{\mathbb{C}U^\tau}$.

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